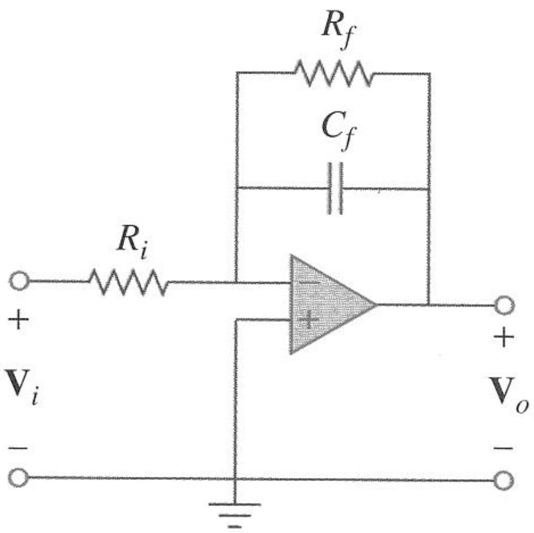
ELG3336: Op Amp-based Active Filters

- **Advantages:**
  - Reduced size and weight, and therefore parasitics.
  - Increased reliability and improved performance.
  - Simpler design than for passive filters and can realize a wider range of functions as well as providing voltage gain.
  - In large quantities, the cost of an IC is less than its passive counterpart.
- **Disadvantages:**
  - Limited bandwidth of active devices limits the highest attainable pole frequency and therefore applications above 100 kHz (passive RLC filters can be used up to 500 MHz).
  - The achievable quality factor is also limited.
  - Require power supplies (unlike passive filters).
  - Increased sensitivity to variations in circuit parameters caused by environmental changes compared to passive filters.
- For applications, particularly in voice and data communications, the economic and performance advantages of active RC filters far outweigh their disadvantages.

First-Order Low-Pass Filter



$$H(f) = \frac{V_o}{V_i} = -\frac{Z_f}{Z_i}$$

$$\frac{1}{Z_f} = \frac{1}{R_f} + \frac{1}{\frac{1}{j2\pi f C_f}} = \frac{1}{R_f} + \frac{j2\pi f R_f C_f}{R_f}$$

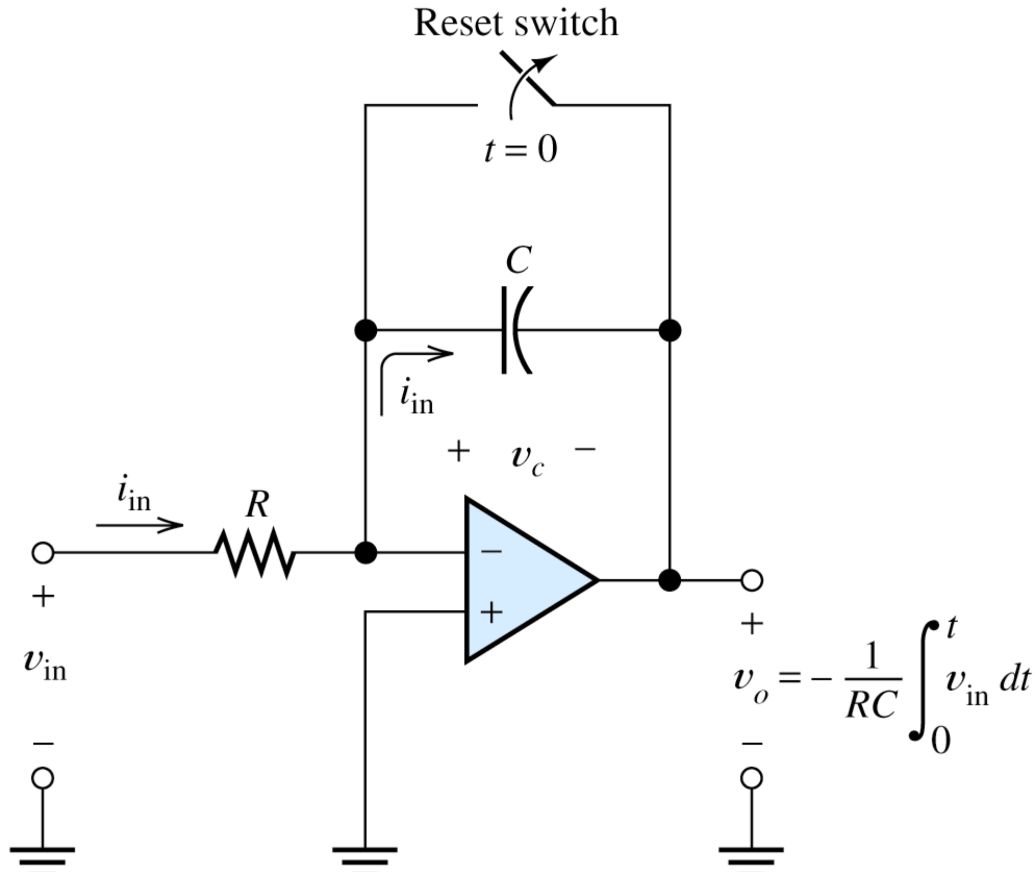
$$Z_f = \frac{R_f}{1 + j2\pi f R_f C_f}$$

$$H(f) = -\frac{Z_f}{Z_i} = -\left(\frac{R_f}{R_i}\right)\frac{1}{1 + j2\pi f R_f C_f}$$

$$= -\left(\frac{R_f}{R_i}\right)\left[\frac{1}{1 + j(f / f_B)}\right]$$

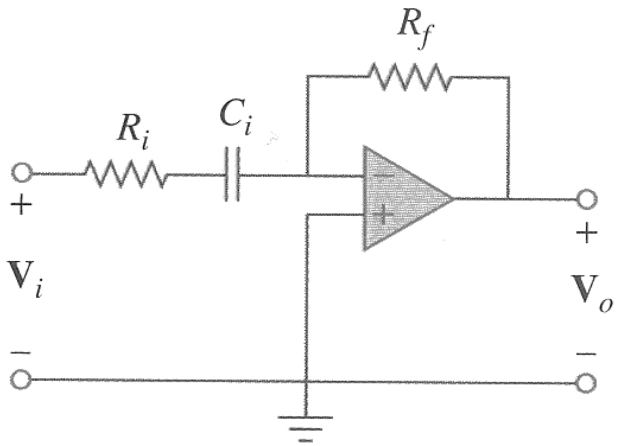
$$f_B = \frac{1}{2\pi R_f C_f}$$

A low-pass filter with a dc gain of $-R_f/R_i$



$$v_o(t) = -\frac{1}{RC}\int_0^t v_{in}(t)\,dt$$

First-Order High-Pass Filter



$$H(f) = \frac{v_o}{v_i} = -\frac{Z_f}{Z_i}$$

$$Z_i = R_i + \frac{1}{j2\pi f C_i} \quad Z_f = R_f$$

$$H(f) = -\frac{Z_f}{Z_i} = -\frac{R_f}{R_i + \frac{1}{j2\pi f C_i}}$$

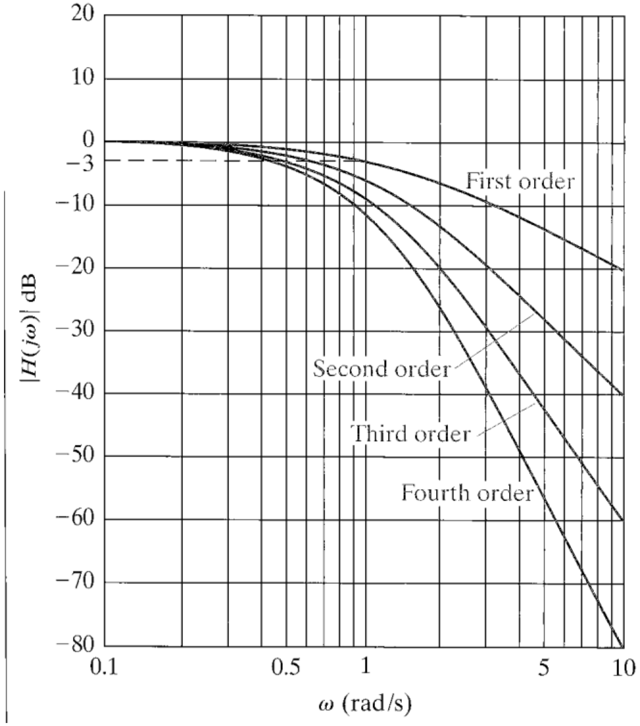
$$= -\frac{j2\pi f R_f C_i}{1 + j2\pi f R_i C_i} = -\left(\frac{R_i}{R_i}\right)\frac{j2\pi f R_f C_i}{1 + j2\pi f R_i C_i}$$

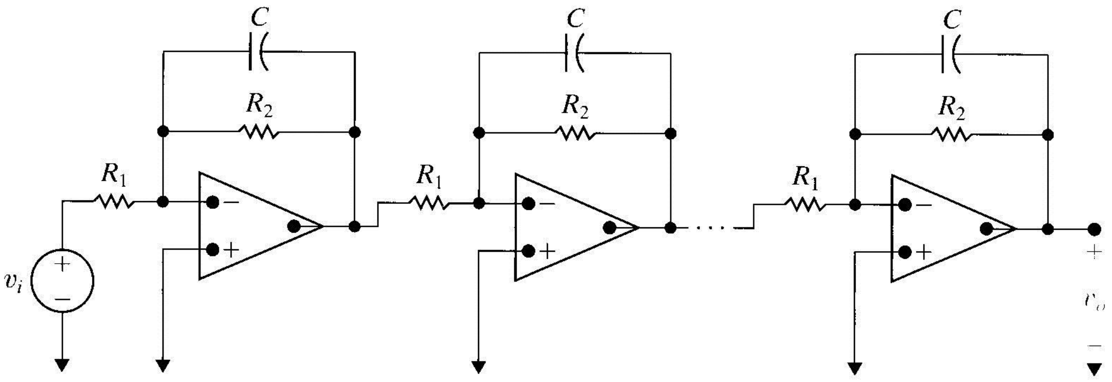
$$= -\left(\frac{R_f}{R_i}\right)\left[\frac{j(f / f_B)}{1 + j(f / f_B)}\right]$$

$$f_B = \frac{1}{2\pi R_i C_i}$$

A high-pass filter with a high frequency gain of $-R_f/R_i$

Higher Order Filters



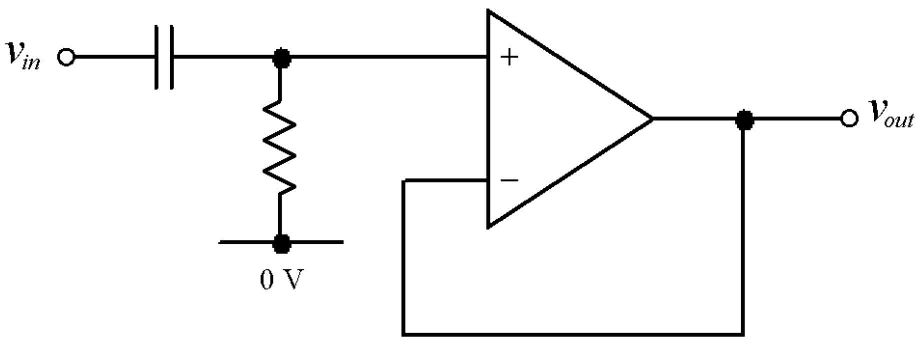


$$H(f) = H_1(f)H_2(f)***H_n(f)$$

$$= (-1)^n \left(\frac{R_f}{R_i}\right)^n \left[\frac{1}{1 + j(f/f_B)}\right]^n$$

Single-Pole Active Filter Designs

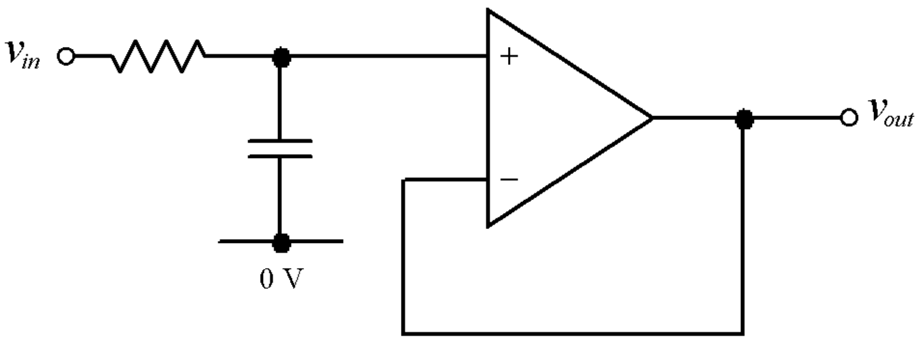
High Pass



$$\frac{v_{out}}{v_{in}} = \frac{1}{\dfrac{1}{sCR}+1} = \frac{1}{\dfrac{1+sRC}{sCR}}$$

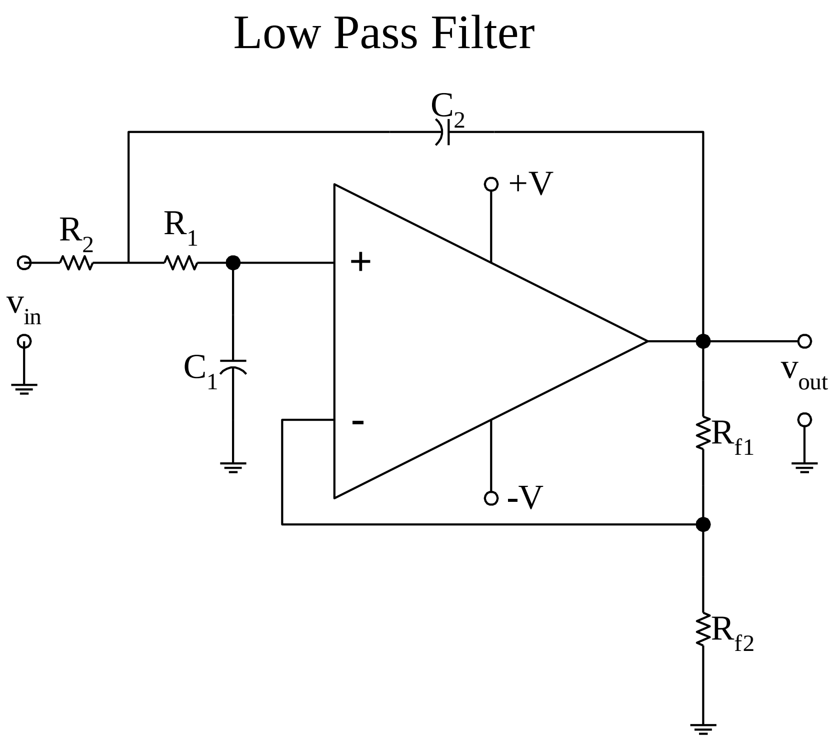
$$= \frac{sRC}{RC(s+1/RC)} = \frac{s}{(s+1/RC)}$$

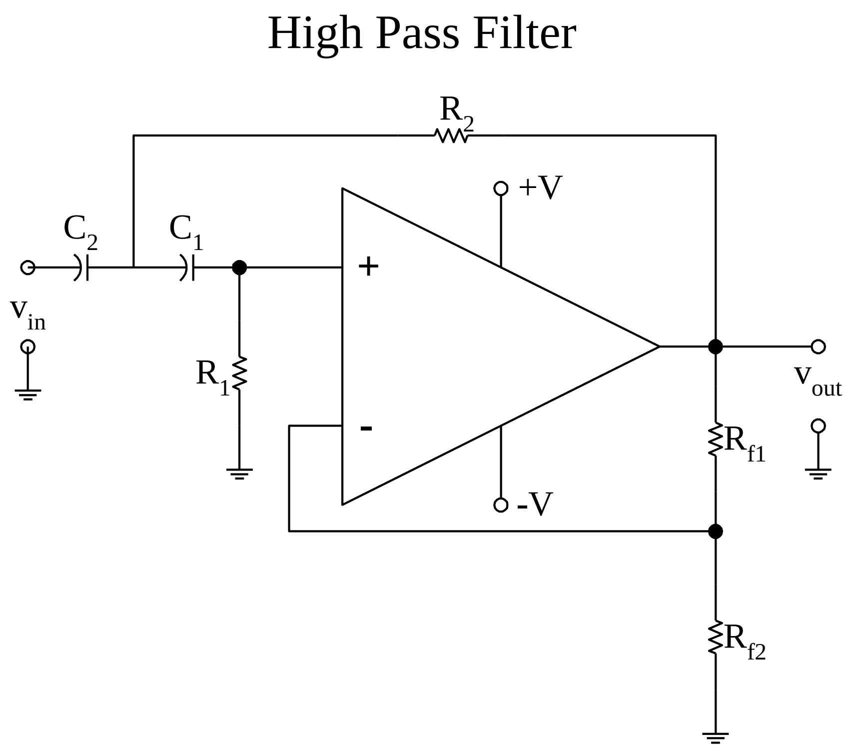
Low Pass



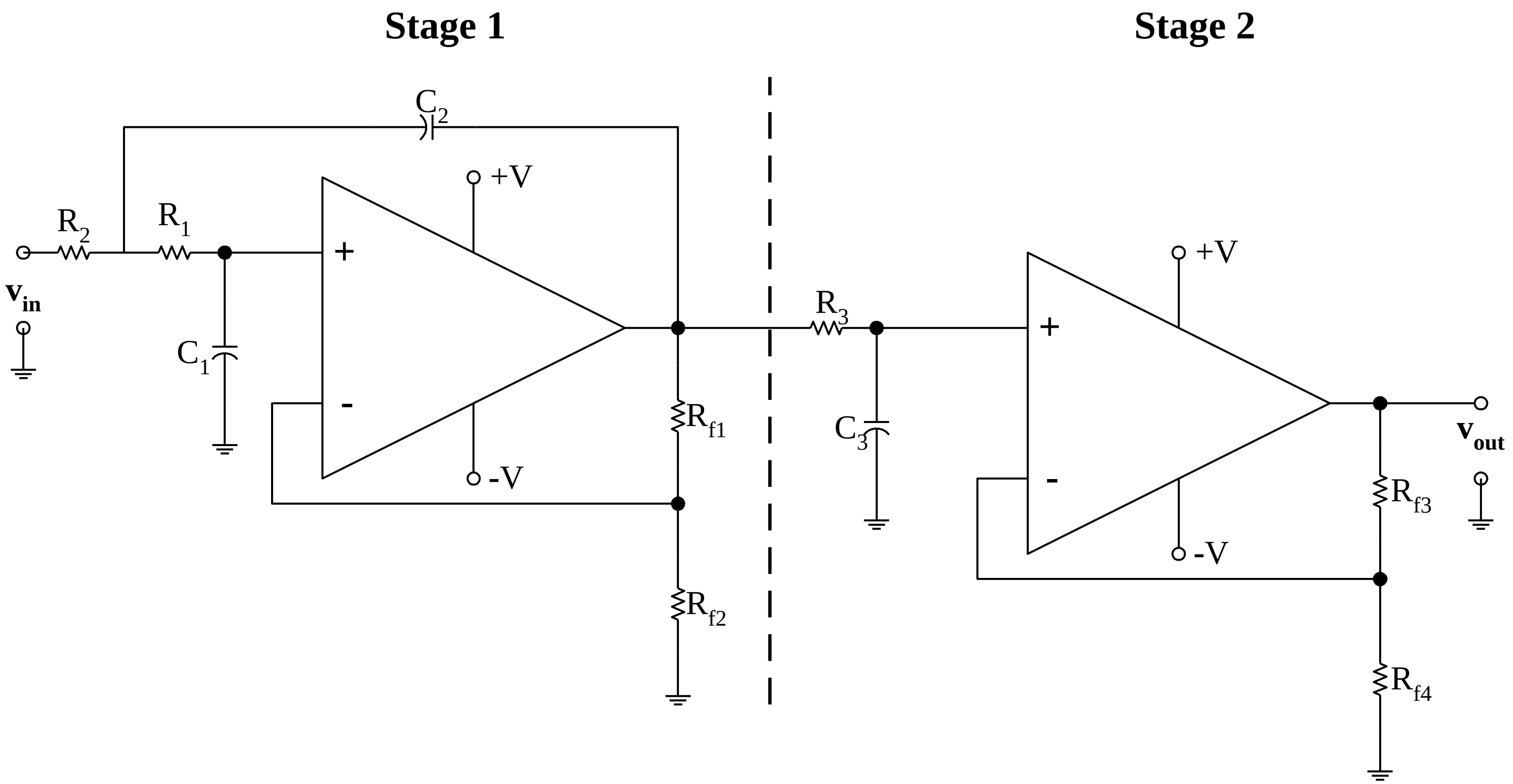
$$\frac{v_{out}}{v_{in}} = \frac{1/RC}{s+1/RC}$$

Two-Pole (Sallen-Key) Filters

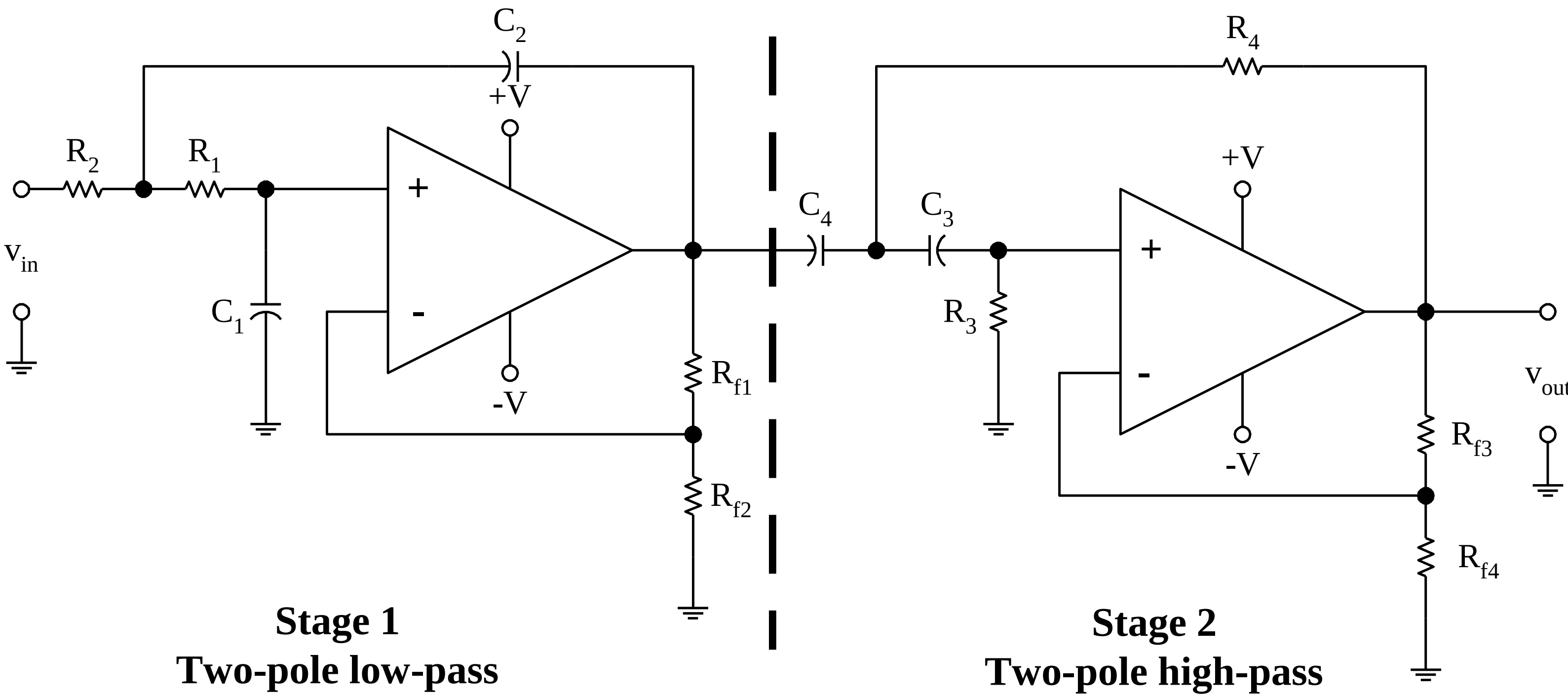


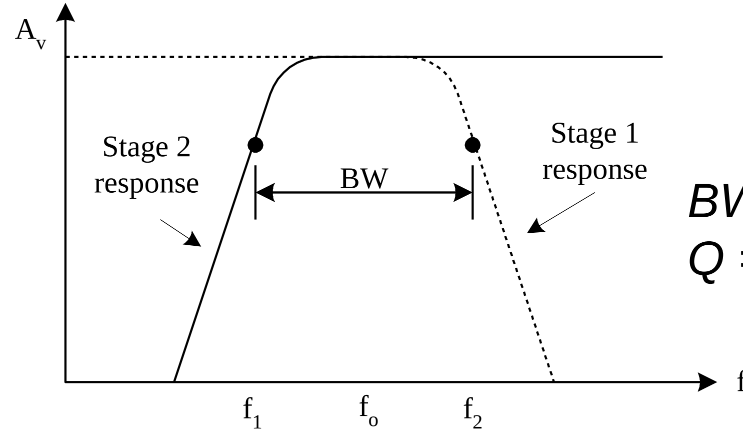


Three-Pole Low-Pass Filter



Two-Stage Band-Pass Filter

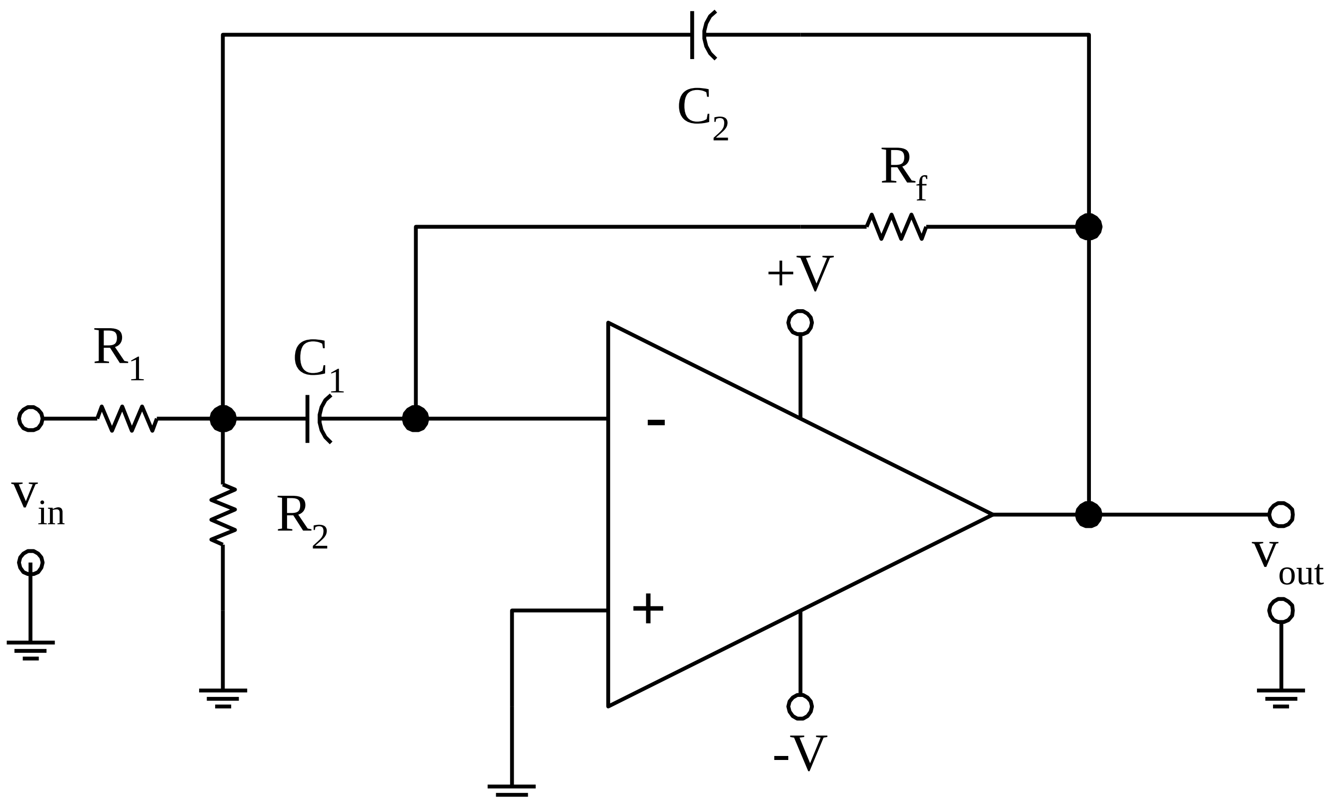




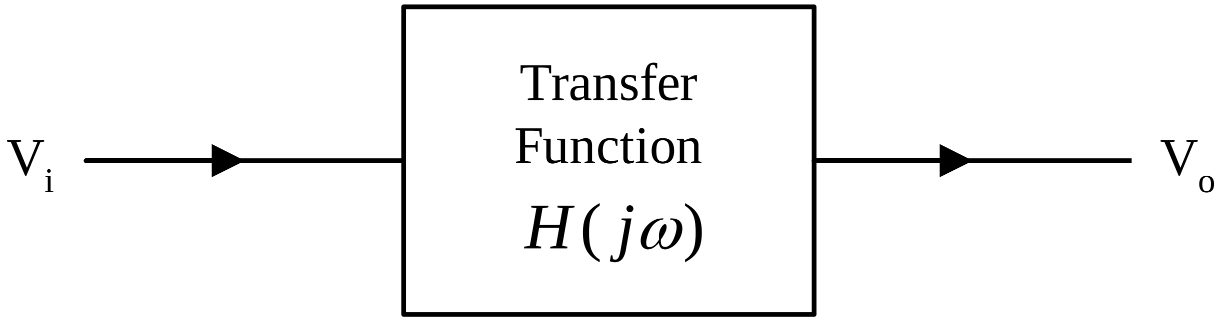
$BW = f_2 - f_1$

$Q = f_0 / BW$

Multiple-Feedback Band-Pass Filter



Transfer function $H(j\omega)$



$$H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)}$$

$$H = \mathrm{Re}(H) + j\,\mathrm{Im}(H)$$

$$|H| = \sqrt{\mathrm{Re}(H)^2 + \mathrm{Im}(H)^2}$$

$$\angle H = \tan^{-1}\left(\frac{\mathrm{Im}(H)}{\mathrm{Re}(H)}\right) \qquad \mathrm{Re}(H) \succ 0$$

$$\angle H = 180^o + \tan^{-1}\left(\frac{\mathrm{Im}(H)}{\mathrm{Re}(H)}\right) \qquad \mathrm{Re}(H) \prec 0$$

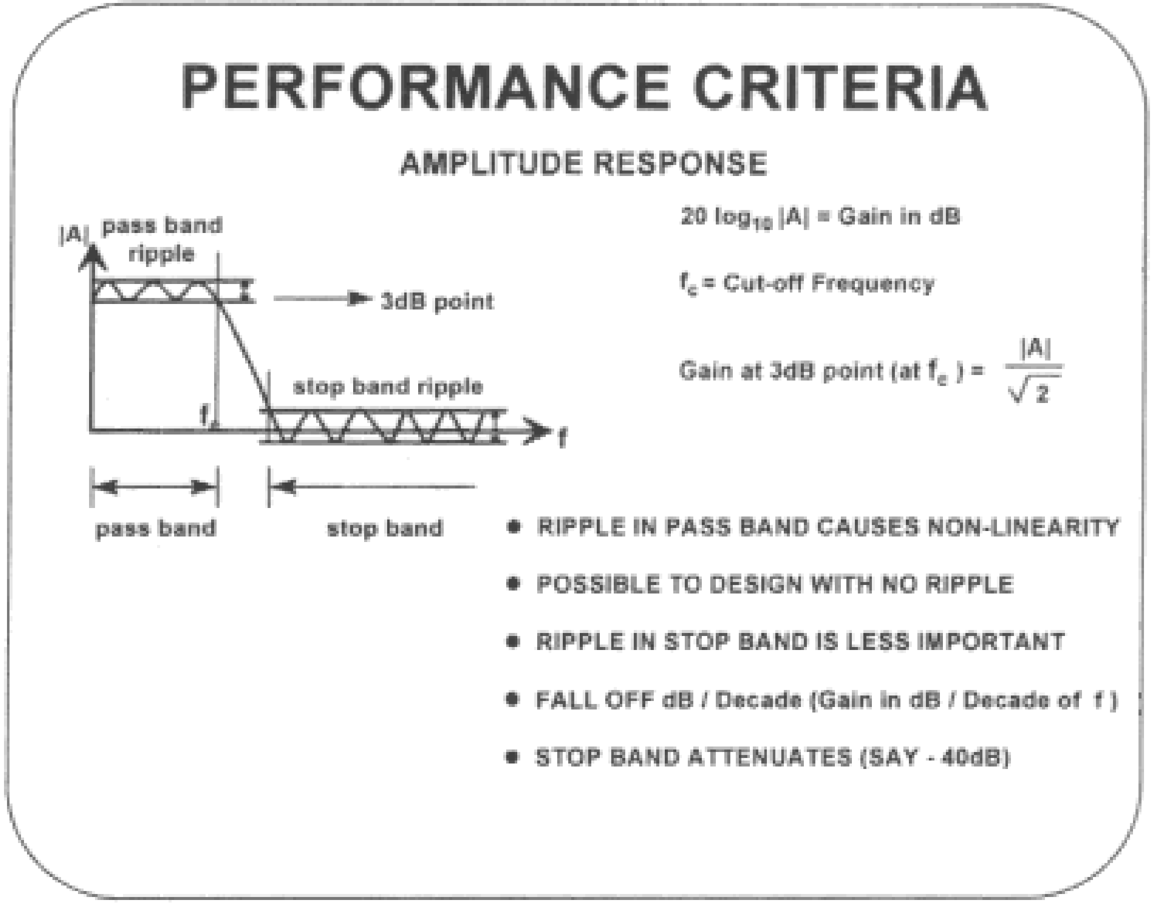
Corner Frequency

- The significance of the break frequency is that it represents the frequency where

$$A_v(f) = 070.7\angle{-45^\circ}$$

- This is where the output of the transfer function has an amplitude *3-dB* below the input amplitude, and the output phase is shifted by -45° relative to the input.

- Therefore, f_c is also known as the **3-dB frequency** or the **corner frequency**.

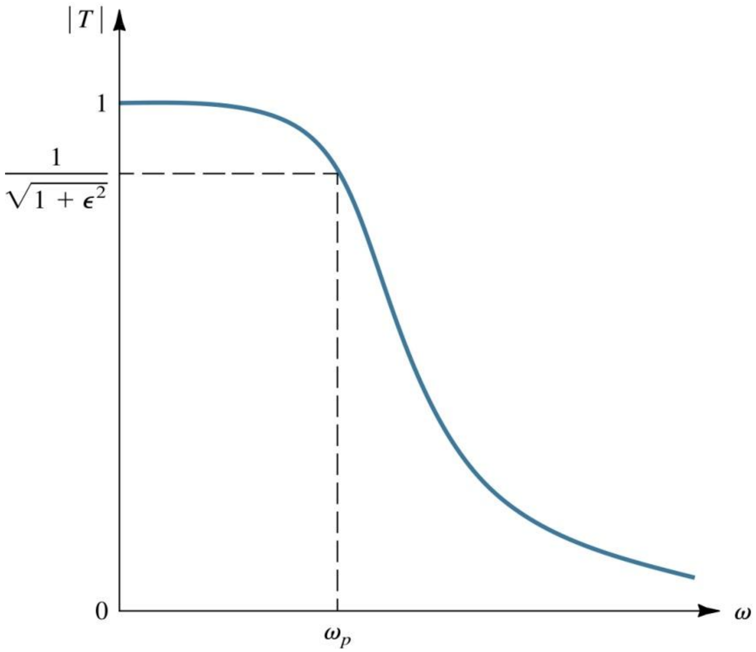


Bode plots use a logarithmic scale for frequency, where a *decade* is defined as a range of frequencies where the highest and lowest frequencies differ by a factor of 10.

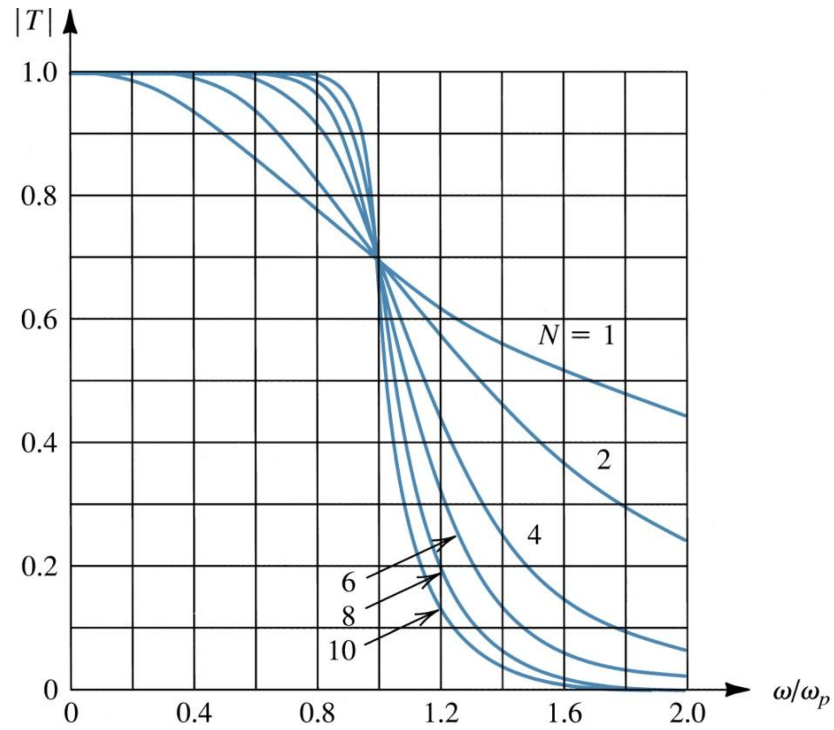
Real Filters

- Butterworth Filters
  - Flat Pass-band.
  - 20*n* dB per decade roll-off.
- Chebyshev Filters
  - Pass-band ripple.
  - Sharper cut-off than Butterworth.
- Elliptic Filters
  - Pass-band and stop-band ripple.
  - Even sharper cut-off.
- Bessel Filters
  - Linear phase response – i.e. no signal distortion in pass-band.

Butterworth Filters

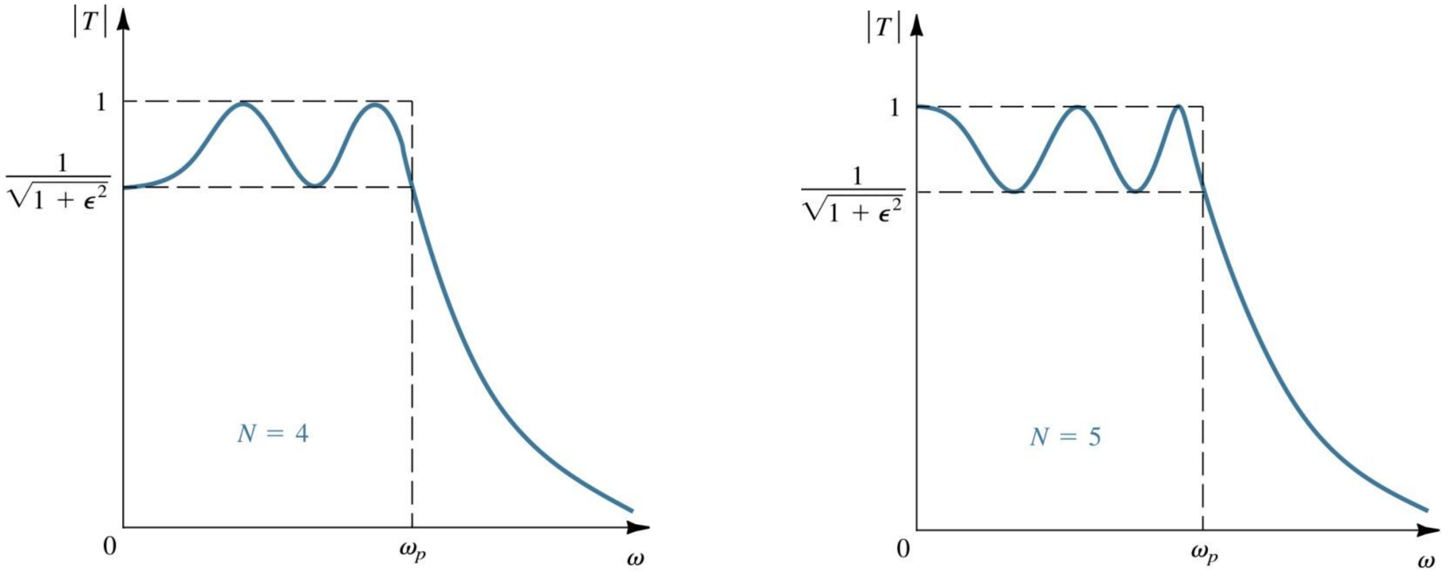


The magnitude response of a Butterworth filter.



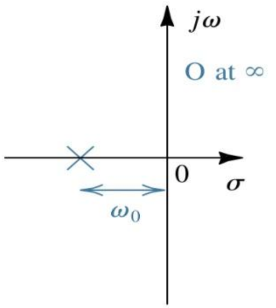
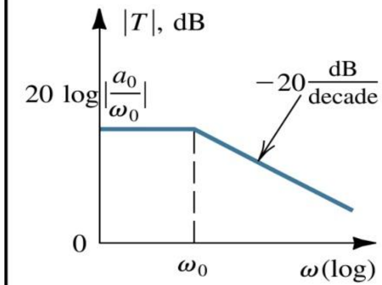
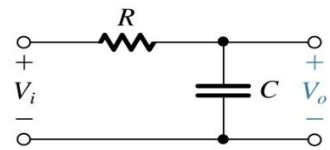
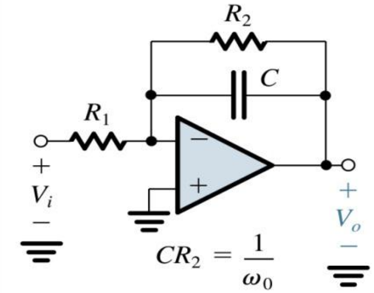
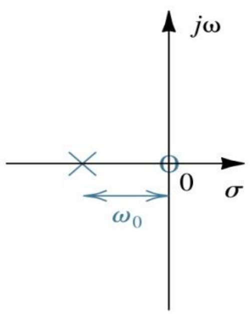
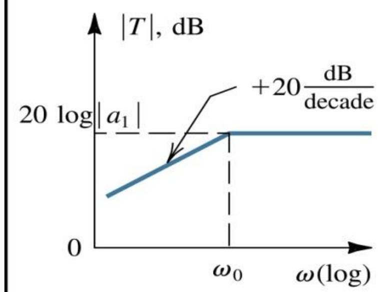
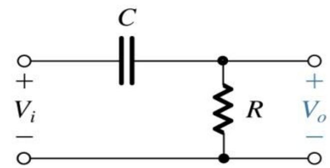
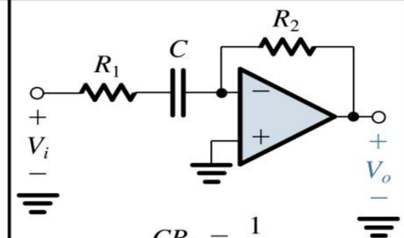
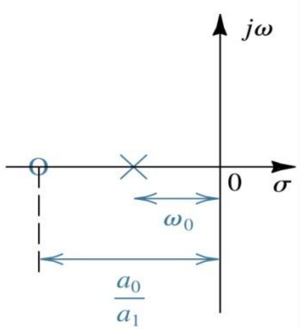
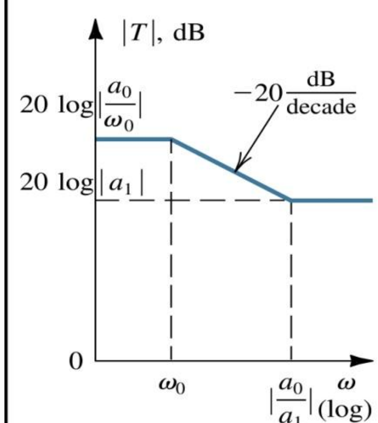
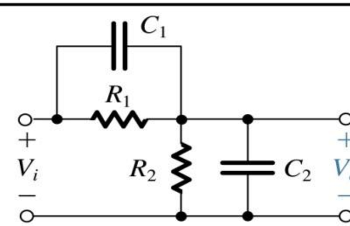
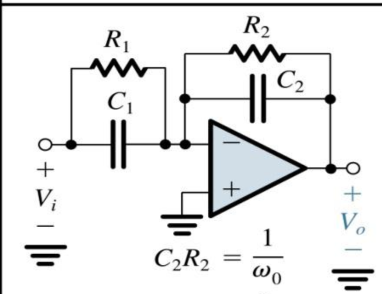
Magnitude response for Butterworth filters of various order with ε = 1. Note that as the order increases, the response approaches the ideal brickwall type transmission.

Chebyshev Filters

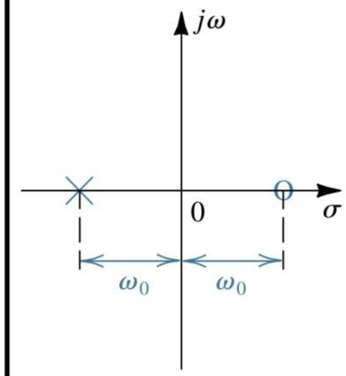
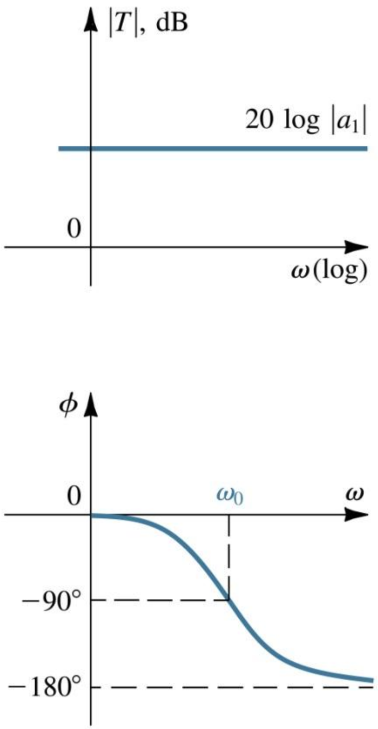
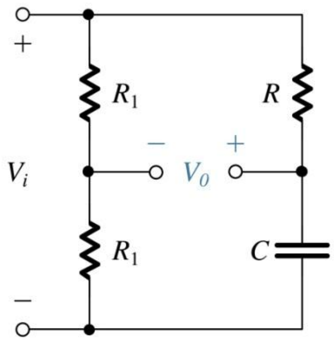
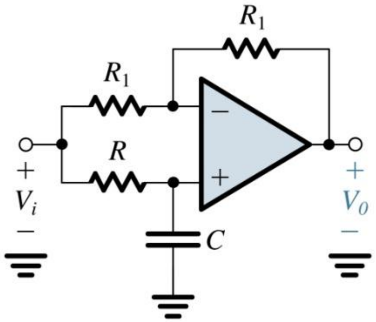


Sketches of the transmission characteristics of a representative even- and odd-order Chebyshev filters.

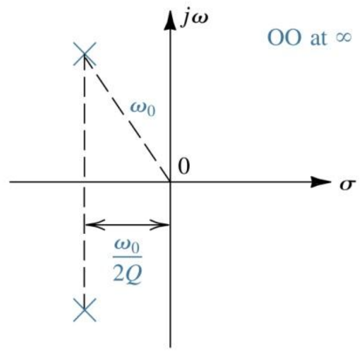
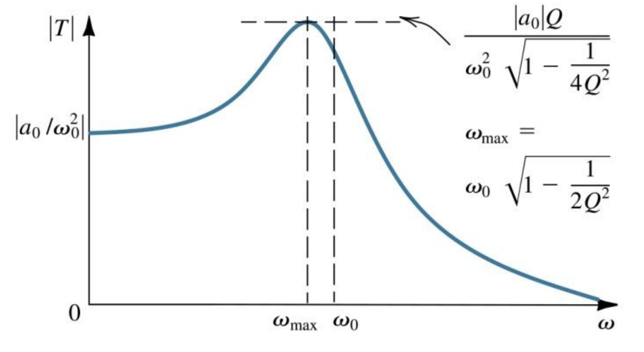
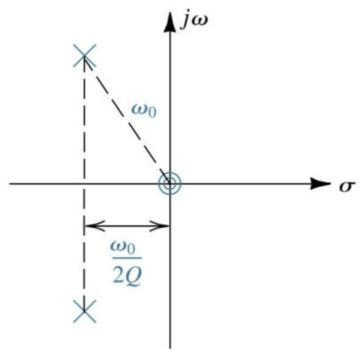
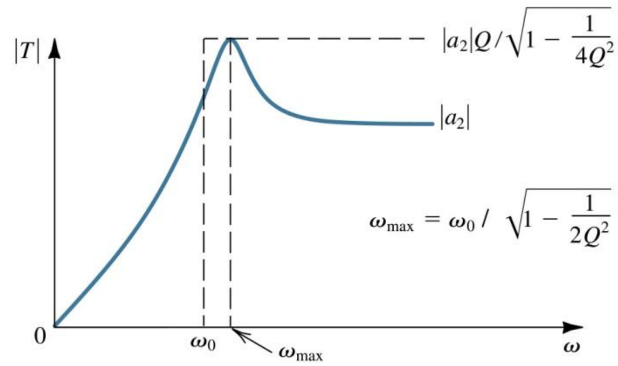
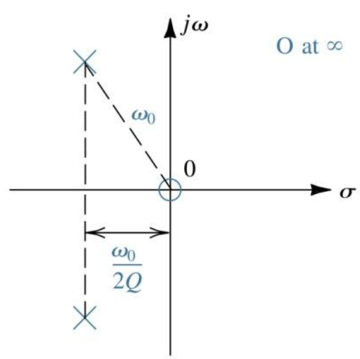
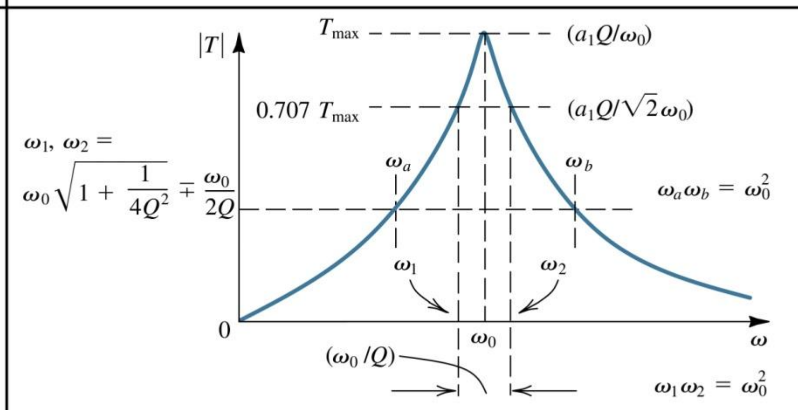
First-Order Filter Functions

Filter Type and $T(s)$	s-Plane Singularities	Bode Plot for $\lvert T \rvert$	Passive Realization	Op Amp-RC Realization
(a) Low-Pass (LP) $T(s) = \dfrac{a_0}{s + \omega_0}$			 $CR = \dfrac{1}{\omega_0}$ dc gain $= 1$	 $CR_2 = \dfrac{1}{\omega_0}$ dc gain $= -\dfrac{R_2}{R_1}$
(b) High-Pass (HP) $T(s) = \dfrac{a_1 s}{s + \omega_0}$			 $CR = \dfrac{1}{\omega_0}$ High-frequency gain $= 1$	 $CR_1 = \dfrac{1}{\omega_0}$ High-frequency gain $= -\dfrac{R_2}{R_1}$
(c) General $T(s) = \dfrac{a_1 s + a_0}{s + \omega_0}$			 $(C_1 + C_2)(R_1 // R_2) = \dfrac{1}{\omega_0}$ $C_1 R_1 = \dfrac{a_0}{a_1}$ dc gain $= \dfrac{R_2}{R_1 + R_2}$ HF gain $= \dfrac{C_1}{C_1 + C_2}$	 $C_2 R_2 = \dfrac{1}{\omega_0}$ $C_1 R_1 = \dfrac{a_1}{a_0}$ dc gain $= -\dfrac{R_2}{R_1}$ HF gain $= -\dfrac{C_1}{C_2}$

First-Order Filter Functions

$T(s)$	Singularities	$\lvert T\rvert$ and ϕ	Passive Realization	Op Amp-RC Realization
$T(s) = -a_1 \frac{s - \omega_0}{s + \omega_0}$ $a_1 > 0$			 $CR = 1/\omega_0$ Flat gain $(a_1) = 0.5$	 $CR = 1/\omega_0$ Flat gain $(a_1) = 1$

Second-Order Filter Functions

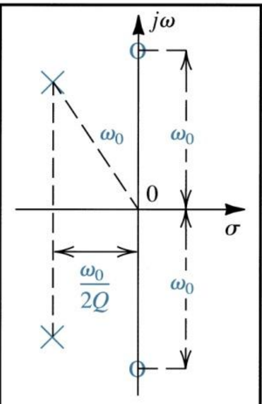
Filter Type and $T(s)$	s-Plane Singularities	$\lvert T\rvert$
(a) Low-Pass (LP) $T(s) = \dfrac{a_0}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$ dc gain $= \dfrac{a_0}{\omega_0^2}$		
(b) High-Pass (HP) $T(s) = \dfrac{a_2 s^2}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$ High-frequency gain $= a_2$		
(c) Bandpass (BP) $T(s) = \dfrac{a_1 s}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$ Center-frequency gain $= \dfrac{a_1 Q}{\omega_0}$		

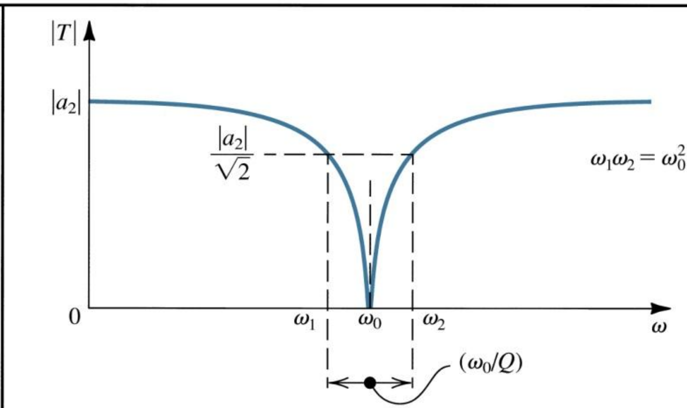
Second-Order Filter Functions

(d) Notch

$$T(s) = a_2 \frac{s^2 + \omega_0^2}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$$

dc gain = high-frequency gain = a_2





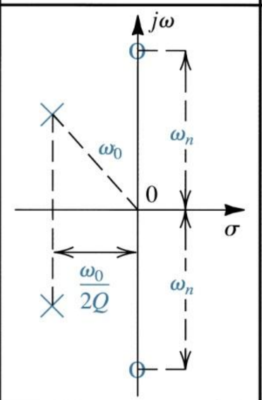
(e) Low-Pass Notch (LPN)

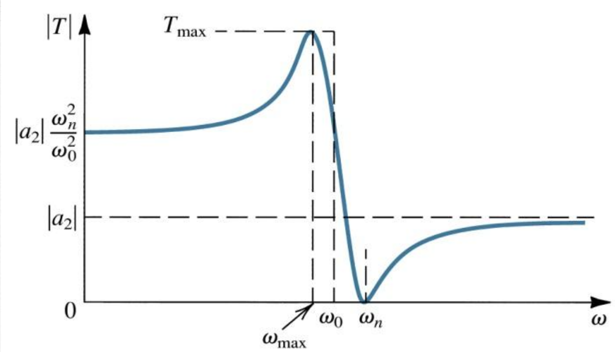
$$T(s) = a_2 \frac{s^2 + \omega_n^2}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$$

$\omega_n \geqslant \omega_0$

dc gain = $a_2 \dfrac{\omega_n^2}{\omega_0^2}$

high-frequency gain = a_2





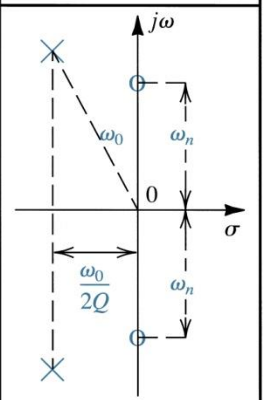
(f) High-Pass Notch (HPN)

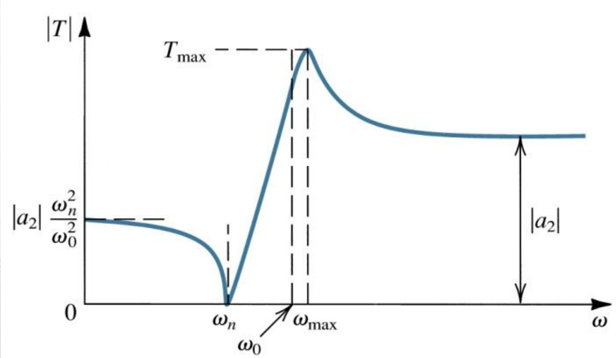
$$T(s) = a_2 \frac{s^2 + \omega_n^2}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$$

$\omega_n \leqslant \omega_0$

dc gain = $a_2 \dfrac{\omega_n^2}{\omega_0^2}$

high-frequency gain = a_2





$$\omega_{max} = \omega_0 \sqrt{\frac{\left(\dfrac{\omega_n^2}{\omega_0^2}\right)\left(1 - \dfrac{1}{2Q^2}\right) - 1}{\dfrac{\omega_n^2}{\omega_0^2} + \dfrac{1}{2Q^2} - 1}}$$

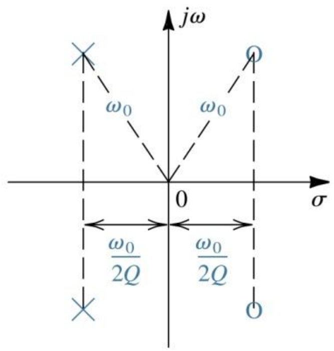
$$T_{max} = \frac{|a_2|\,|\omega_n^2 - \omega_{max}^2|}{\sqrt{(\omega_0^2 - \omega_{max}^2)^2 + \left(\dfrac{\omega_0}{Q}\right)^2 \omega_{max}^2}}$$

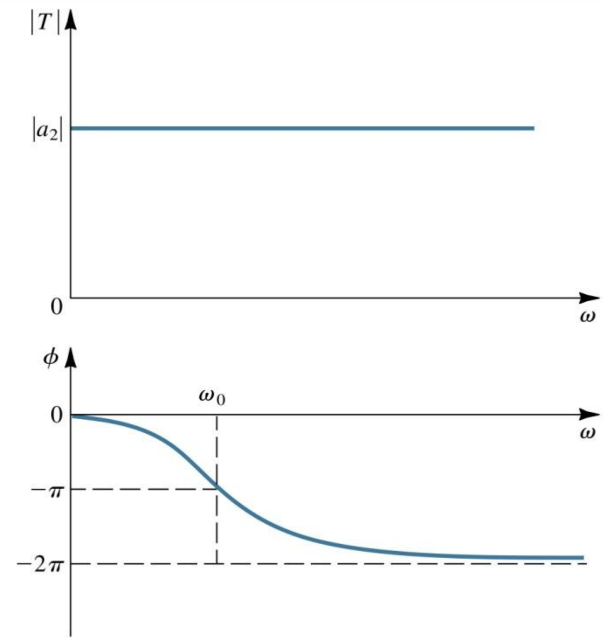
Second-Order Filter Functions

(g) All-Pass
(AP)

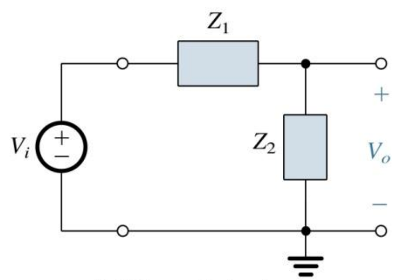
$$T(s) = a_2 \frac{s^2 - s\dfrac{\omega_0}{Q} + \omega_0^2}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$$

Flat gain $= a_2$

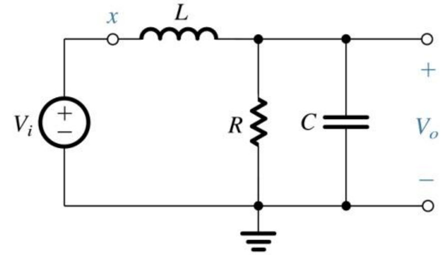




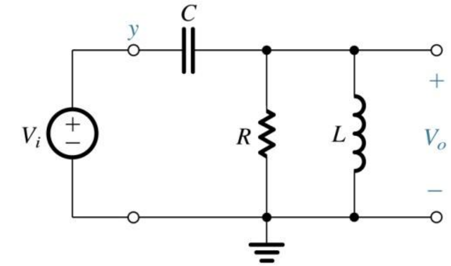
Second-Order LCR Resonator



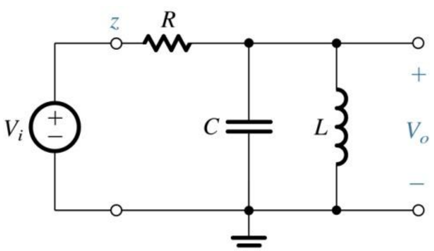
(a) General structure



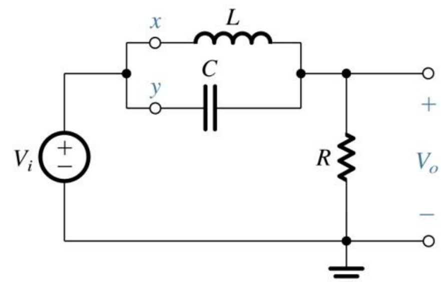
(b) LP



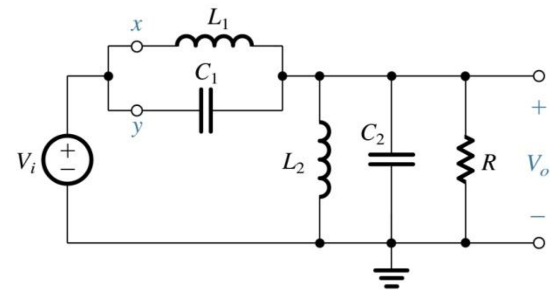
(c) HP



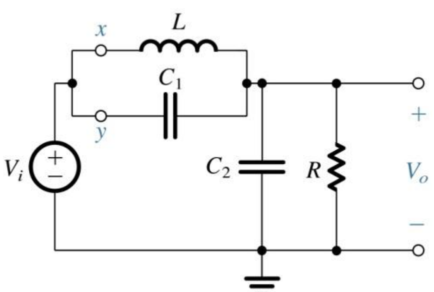
(d) BP



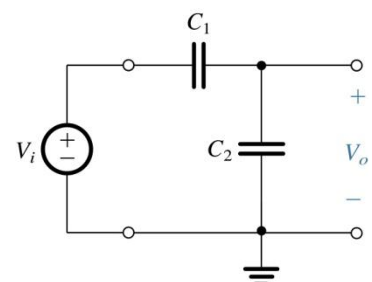
(e) Notch at ω_0



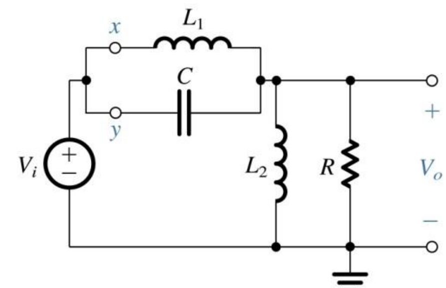
(f) General notch



(g) LPN $(\omega_n > \omega_0)$

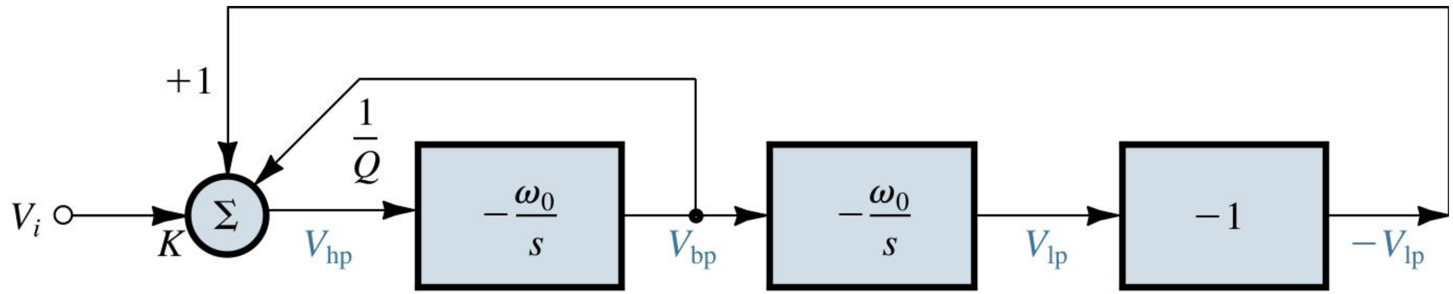


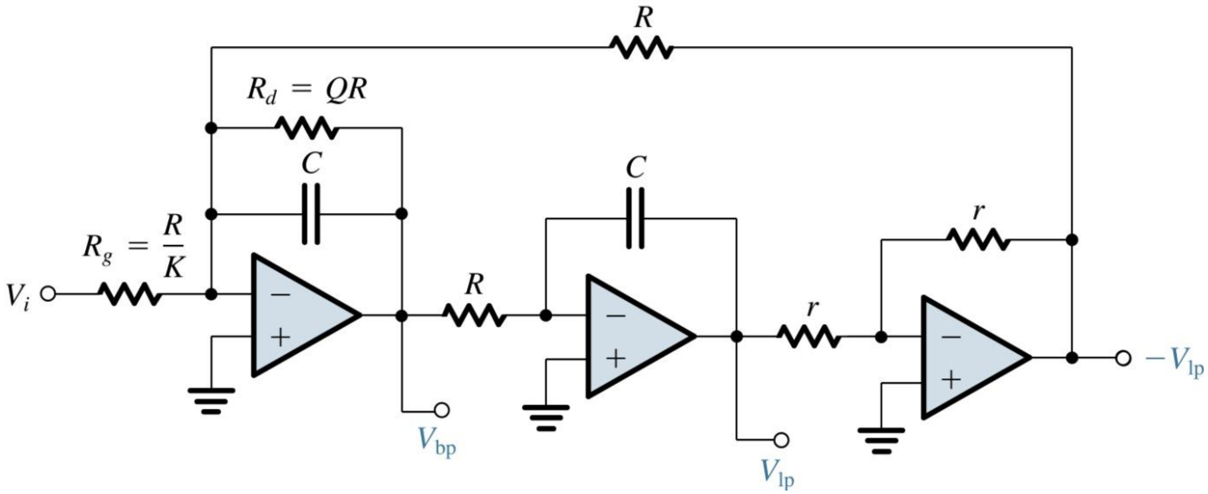
(h) LPN as $s \to \infty$



(i) HPN $(\omega_n < \omega_0)$

Second-Order Active Filter: Two-Integrator-Loop

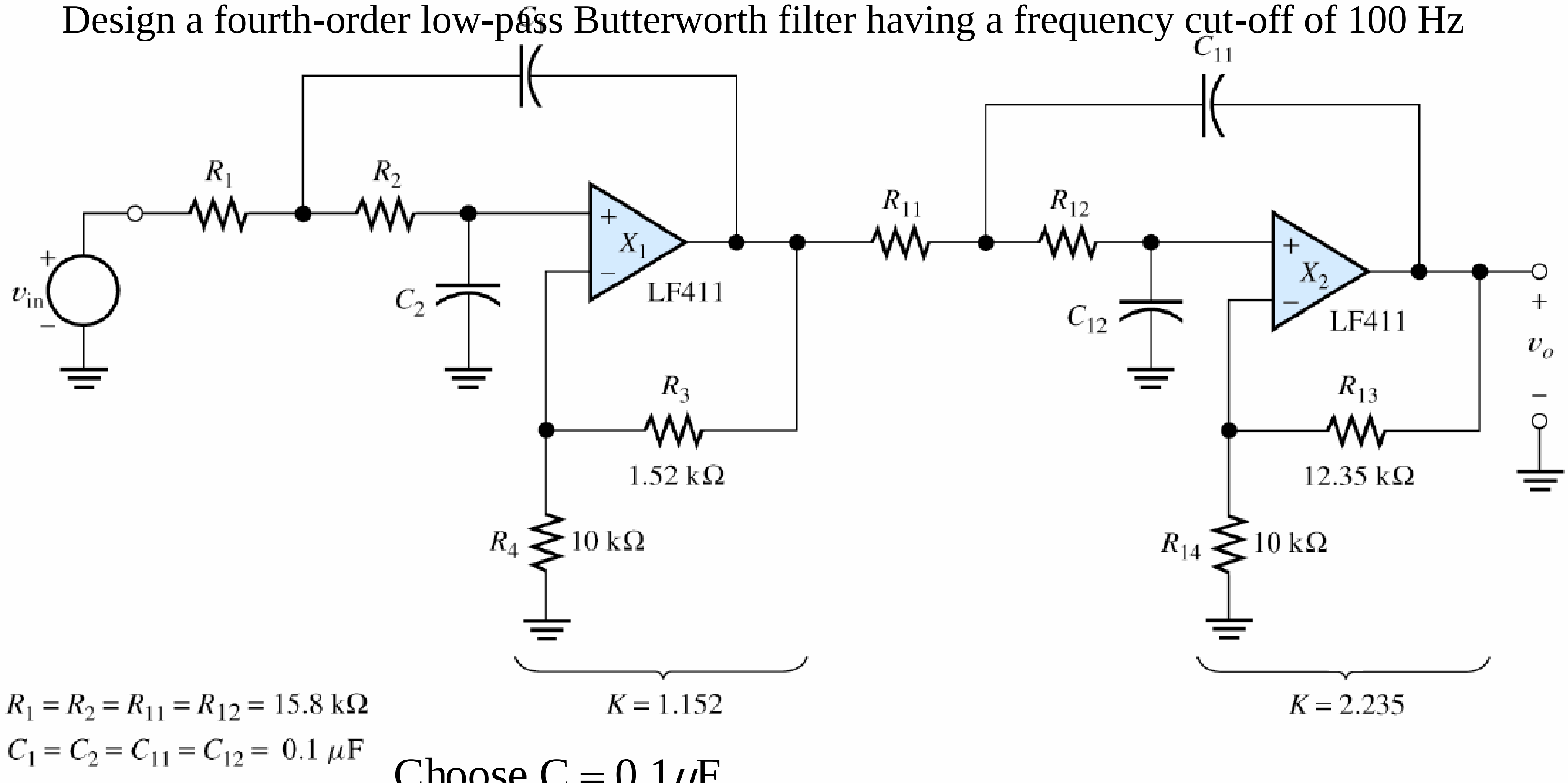




Derivation of an alternative two-integrator-loop biquad in which all op amps are used in a single-ended fashion. The resulting circuit in (b) is known as the Tow-Thomas biquad.

Low-Pass Active Filter Design

Design a fourth-order low-pass Butterworth filter having a frequency cut-off of 100 Hz

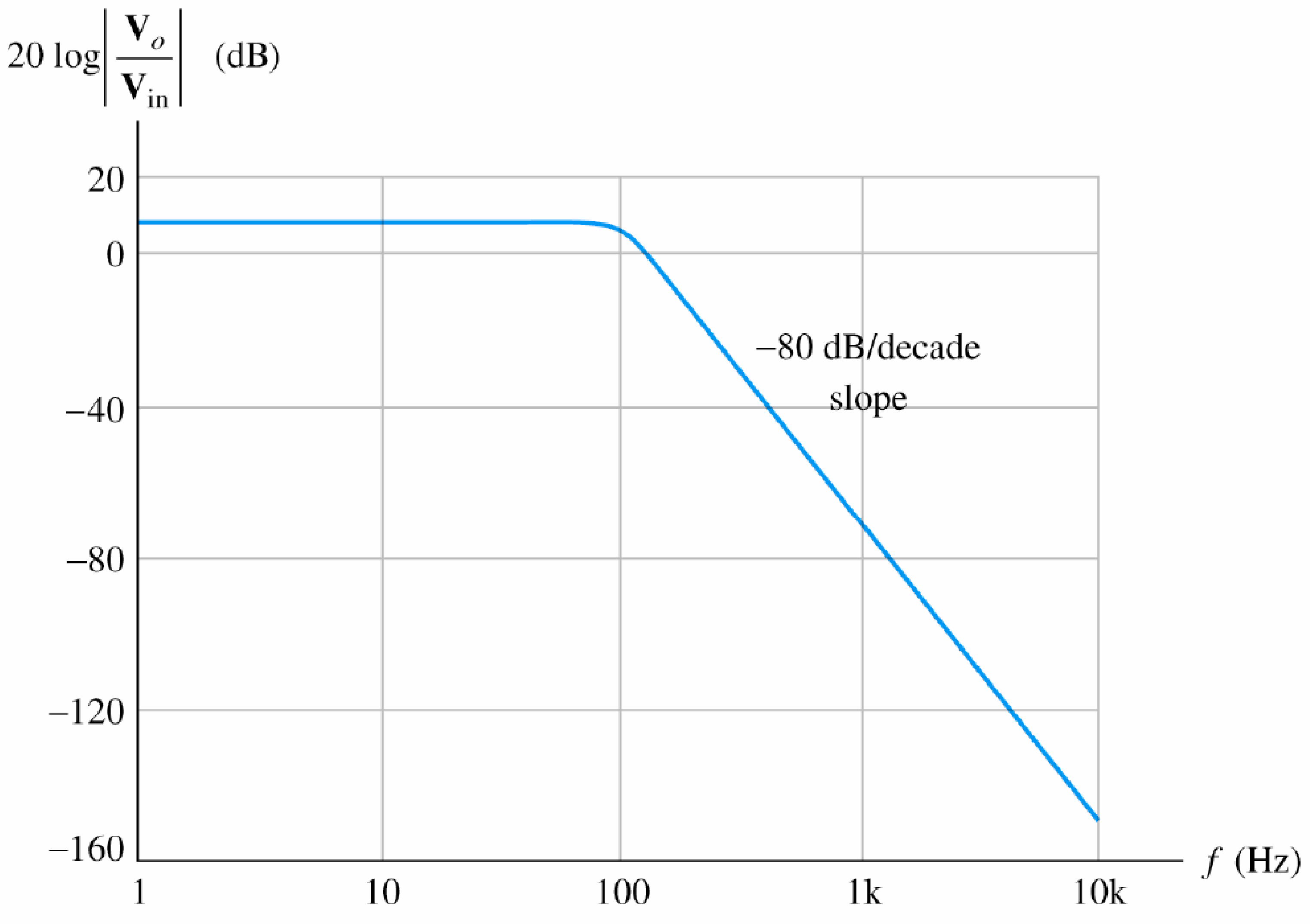


Choose $\mathrm{C} = 0.1\mu\mathrm{F}$

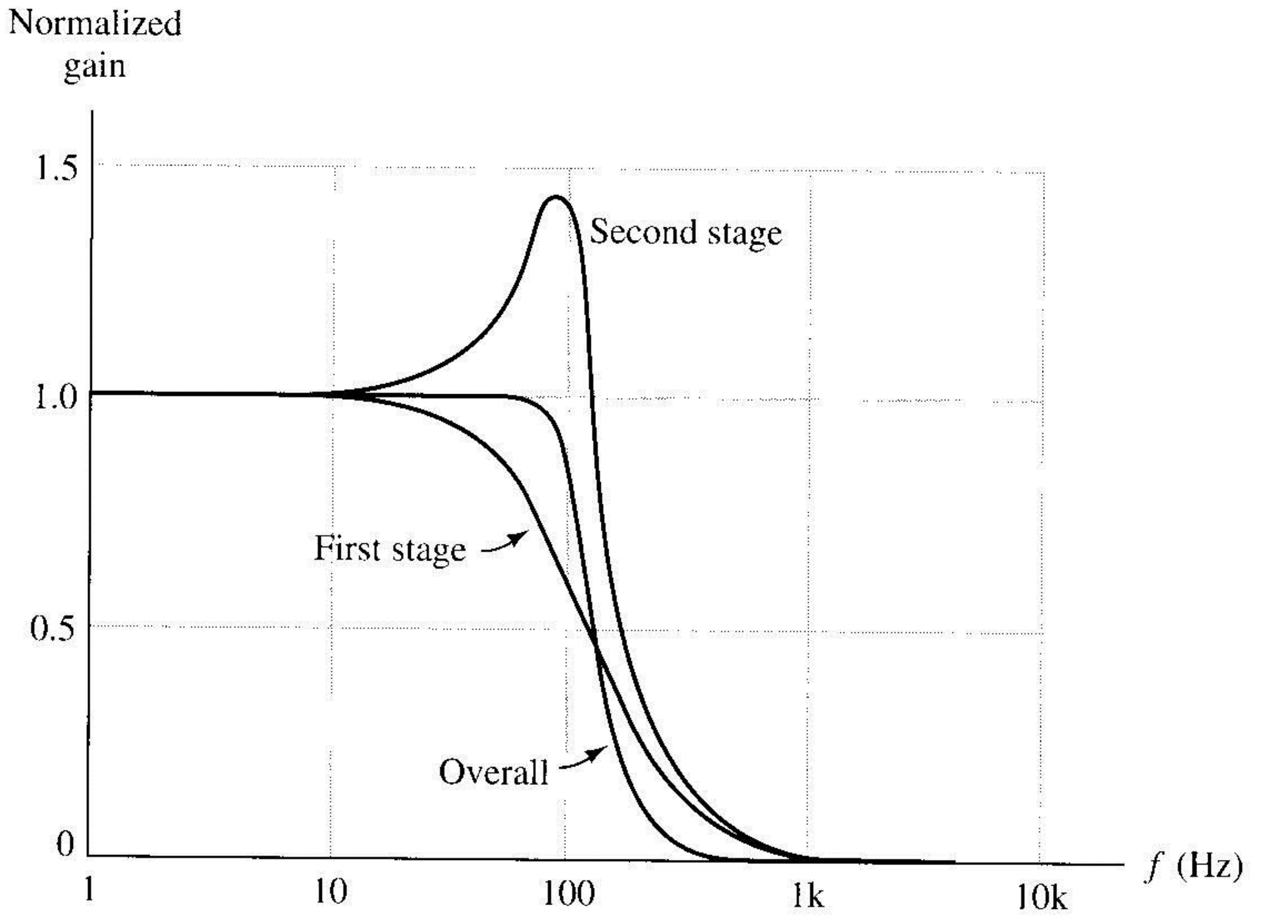
$$R = \frac{1}{2\pi C f_B} = \frac{1}{2\pi(0.1x10^{-6})(100Hz)}$$

$$= 15.92k\Omega$$

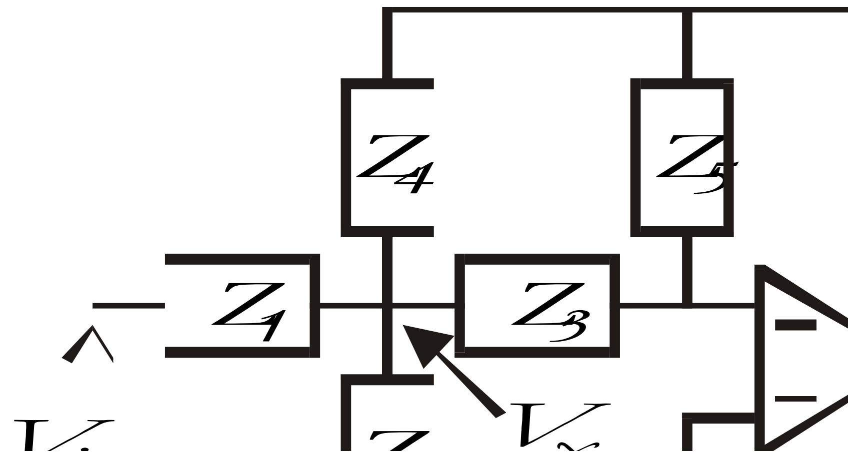
Low-Pass Active Filter Design



Low-Pass Active Filter Design



Infinite-Gain Multiple-Feedback (IGMF) Negative Feedback Active Filter



$$v_i^+ = 0 \quad v_i^- = 0 \quad \Rightarrow \quad V_o = -\frac{Z_5}{Z_3} V_x \Rightarrow V_x = -\frac{Z_3}{Z_5} V_o \qquad (1)$$

By KCL at V_x,
$$\frac{V_i - V_x}{Z_1} = \frac{V_x}{Z_2} + \frac{V_x}{Z_3} + \frac{V_x - V_o}{Z_4} \qquad (2)$$

Substitute (1) into (2) gives

$$\frac{V_i}{Z_1} + \frac{Z_3}{Z_1 Z_5} V_o = -\frac{Z_3}{Z_5 Z_2} V_o - \frac{V_o}{Z_5} - \frac{Z_3}{Z_4 Z_5} V_o - \frac{V_o}{Z_4} \qquad (3)$$

rearranging equation (3), it gives,

$$H = \frac{V_o}{V_i} = -\frac{\frac{1}{Z_1 Z_3}}{\frac{1}{Z_5}\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4}\right) + \frac{1}{Z_3 Z_4}}$$

Or in admittance form:

$$H = \frac{V_o}{V_i} = -\frac{Y_1 Y_3}{Y_5\left(Y_1 + Y_2 + Y_3 + Y_4\right) + Y_3 Y_4}$$

Filter \ Value	Z_1	Z_2	Z_3	Z_4	Z_5
LP	R_1	C_2	R_3	R_4	C_5
HP	C_1	R_2	C_3	C_4	R_5
BP	R_1	R_2	C_3	C_4	R_5

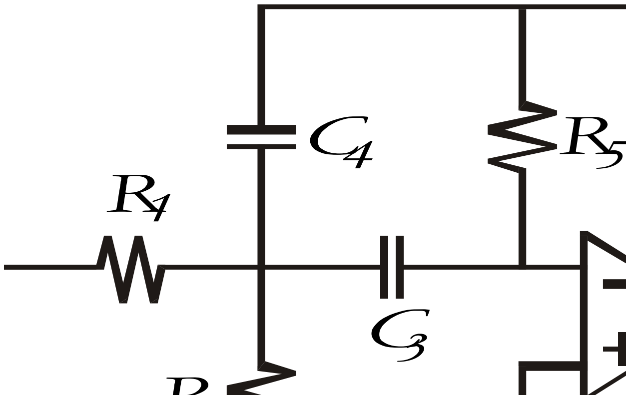
IGMF Band-Pass Filter

Band-pass: $$H(s) = K\frac{s}{s^2 + as + b}$$

To obtain the band-pass response, we let

$$Z_1 = R_1 \quad Z_2 = R_2 \quad Z_3 = \frac{1}{j\omega C_3} = \frac{1}{sC_3} \quad Z_4 = \frac{1}{j\omega C_4} = \frac{1}{sC_4} \quad Z_5 = R_5$$

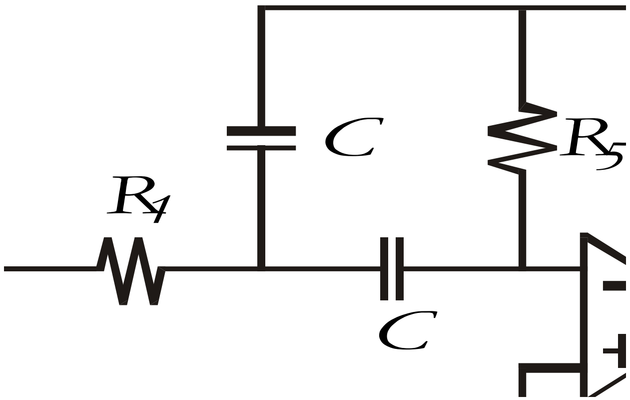
$$H(s) = -\frac{\dfrac{sC_3}{R_1}}{s^2 C_3 C_4 + s\dfrac{C_3 + C_4}{R_5} + \dfrac{1}{R_5}\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)}$$



This filter prototype has a very low sensitivity to component tolerance when compared with other prototypes.

Simplified Design (IGMF Filter)

$$H(s) = -\frac{\frac{sC}{R_1}}{\frac{1}{R_1 R_5} + s\frac{2C}{R_5} + s^2 C^2}$$



Comparing with the band-pass response

$$H(s) = K\frac{s}{s^2 + \frac{\omega_P}{Q_P}s + \omega_P^2}$$

It gives,

$$\omega_p = \frac{1}{C\sqrt{R_1 R_5}} \quad Q_p = \frac{1}{2}\sqrt{\frac{R_5}{R_1}} \quad H(\omega_p) = -2Q^2$$

Example: IGMF Band Pass Filter

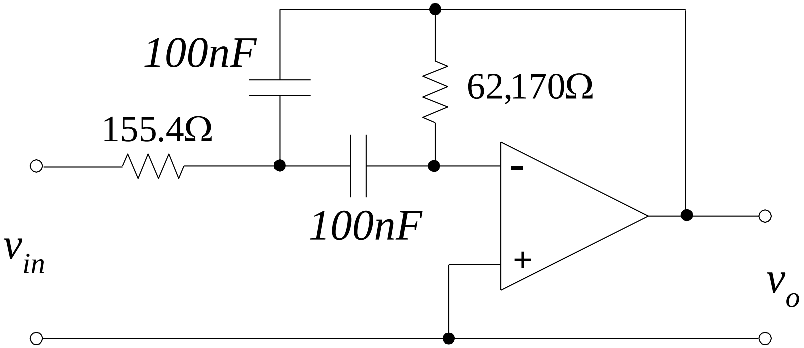
To design a band-pass filter with $f_O = 512Hz$ and $Q = 10$

$$\omega_P = \frac{1}{C\sqrt{R_1 R_5}} = 2\pi(512Hz)$$

$$C = 100\,nF \quad \rightarrow \quad R_1 R_5 = 9{,}662{,}741\Omega^2$$

$$Q_P = \frac{1}{2}\frac{\sqrt{R_5}}{\sqrt{R_1}} = 10$$

$$\rightarrow R_1 = 155.4\Omega \qquad R_5 = 62{,}170\Omega$$



With similar analysis, we can choose the following values:

$$C = 10\,nF \quad R_1 = 1{,}554\,\Omega \quad \text{and} \quad R_5 = 621{,}700\,\Omega$$

Butterworth Response (Maximally Flat)

$$\left|H\left(j\omega\right)\right| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_o}\right)^{2n}}}$$

where n is the order

Normalize to ω_o = *1rad/s*

$$\left|\hat{H}(j\omega)\right| = \frac{1}{\sqrt{1+\omega^{2n}}}$$

Butterworth polynomials:

$$\left|\hat{H}(j\omega)\right| = \frac{1}{\left|B_n(j\omega)\right|}$$

Butterworth polynomials

$$B_1\left(s\right) = s+1$$

$$B_2\left(s\right) = s^2+\sqrt{2}s+1$$

$$\begin{aligned} B_3\left(s\right) &= s^3+2s^2+2s+1 \\ &= \left(s+1\right)\left(s^2+s+1\right) \end{aligned}$$

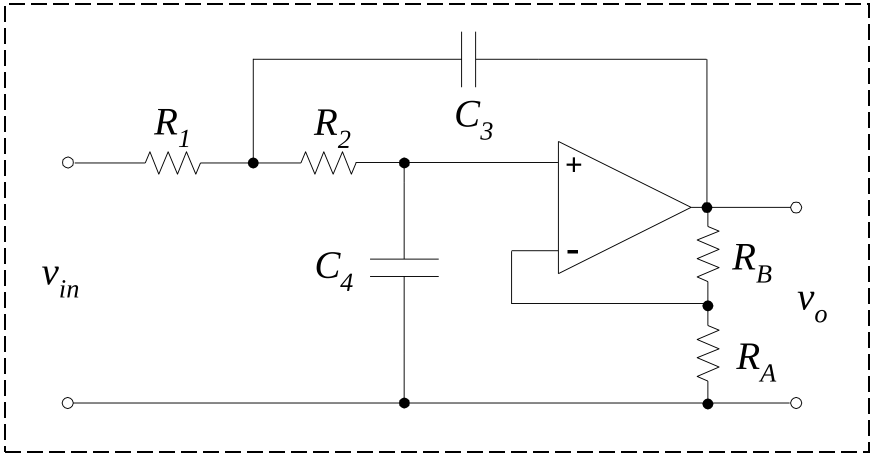
$$\begin{aligned} B_4\left(s\right) &= s^4+2.61s^3+3.41s^2+2.61s+1 \\ &= \left(s^2+0.77s+1\right)\left(s^2+1.85s+1\right) \end{aligned}$$

$$\begin{aligned} B_5\left(s\right) &= s^5+3.24s^4+5.24s^3+5.24s^2+3.24s+1 \\ &= \left(s+1\right)\left(s^2+0.62s+1\right)\left(s^2+1.62s+1\right) \end{aligned}$$

Second order Butterworth Filter

$$H(s)=\frac{K}{1+sC_4(R_1+R_2)+sR_1C_3(1-K)+s^2R_1R_2C_3C_4}=\frac{K'}{s^2+\frac{\omega_P}{Q_P}s+\omega_P^2}$$

$$Q_P=\frac{1}{\sqrt{\frac{R_1C_4}{R_2C_3}}+\sqrt{\frac{R_2C_4}{R_1C_3}}+(1-K)\sqrt{\frac{R_1C_3}{R_2C_4}}}$$



Setting $R_1 = R_2$ and $C_1 = C_2$

$$Q_P=\frac{1}{\sqrt{1}+\sqrt{1}+(1-K)\sqrt{1}}=\frac{1}{2+(1-K)}=\frac{1}{3-K}$$

Now $K = 1 + R_B / R_A$

$$Q_P=\frac{1}{3-K}=\frac{1}{3-\left(1+\frac{R_B}{R_A}\right)}=\frac{1}{2-\frac{R_B}{R_A}}$$

For Butterworth response:

$$Q_P=\frac{1}{\sqrt{2}} \quad \Rightarrow \quad Q_P=\frac{1}{\sqrt{2}}=\frac{1}{2-\frac{R_B}{R_A}}$$

We have $$2-\frac{R_B}{R_A}=\sqrt{2}=1.414$$

We define Damping Factor (DF) as:

$$DF=\frac{1}{Q_P}=2-\frac{R_B}{R_A}=1.414$$