Information Theory & Electromagnetism: Are They Related?

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Introduction

• Extensive progress in information & communication theory and electromagnetic science in last 50 years
• The two areas are completely disconnected
• Doesn’t fit the internal structure: the only carrier of information is the electromagnetic field!
• Prediction: the future integration is inevitable
• Current research is stimulated by MIMO
• This seems to be the closed point between the two areas
• Purpose: not only to answer, but to ask questions!
• May be somewhat speculative
Information Theory

- Random variables/processes, entropy, mutual information, channel capacity
- Fundamental limits on communications
- Birth date: 1948, Shannon’s “Communication in the presence of noise”
- **Basics:**
  - Entropy → average information content of the source per symbol:
    \[
    H(X) = -\sum_{i=1}^{N} p_i \log p_i , \ p_i = \Pr\{x_i\}
    \]
    - It is a measure of uncertainty about x (on average): the more is known about x, the less is the entropy
Information Theory: Basics

- Two or more R.V. -> joint & conditional probabilities -> joint & conditional entropies.
- Joint entropy:
  \[ H(X,Y) = -\sum_{i,j} p(x_i,y_j) \log p(x_i,y_j) \]
- Conditional entropy: the entropy of x given y, averaged over y:
  \[ H(X|Y) = -\sum_{i,j} p(x_i,y_j) \log p(x_i|y_j) \]
- It is a measure of uncertainty about x provided y is known. Since y may provide some information about x,
  \[ H(X|Y) \leq H(X) \]
Information Theory: Basics

• The mutual information:

\[ I(X, Y) = H(X) - H(X|Y) \text{ [bit/symbol]} \]

- \( H(X) \) – measure of uncertainty about \( X \), \( H(X|Y) \) – measure of uncertainty about \( X \) provided we know \( Y \). The difference gives a decrease in uncertainty due to knowledge of \( Y \).

• Channel capacity:

\[ C = \max_{p(x)} I(X, Y) \text{ [bit/symbol]} = \Delta f \log(1 + SNR) \text{ [bit/s]} \]

This is the most fundamental notion in communication & information theory. It gives the fundamental limit on reliable communication over noisy channel.

• **Error-free transmission is possible at** \( R \leq C \) **only!**
Recent Development: MIMO

- Matrix AWGN channel -> celebrated Foschini-Telatar formula:
  \[ C = \log_2 \det \left( I + \frac{\rho}{n} GG^+ \right) \text{[bit/s/Hz]} \]

  \( G \) – normalized channel gain matrix, \( n \) – number of Tx/Rx antennas, \( \rho \) - SNR

- Enormous channel capacity -> 10 fold increase has been demonstrated
- Multipath is not enemy, but ally!
- MIMO channel capacity crucially depends the propagation channel \( G \)
- The impact of electromagnetism comes through \( G \)
MIMO Spectral Efficiency

Capacity, bit/s/Hz

Number of antennas

- MIMO
- convent.
- SISO
How to Avoid Electromagnetism

• If you don’t want to learn it – avoid it!
• Are there many options? (to carry the information)
• Nature provides few of them (fundamental interactions):
  1. Electromagnetism
  2. Gravitation
  3. Strong nuclear force
  4. Weak nuclear force
• The latter two – short range only ($10^{-15}$ & $10^{-18}$ m)
• Long-life particles can be used as well, but difficult to detect,
  – neutrino
How to Avoid Electromagnetism

• Summary:
  – the two fundamental forces are out forever (short range)
  – the gravity is temporarily out: very weak, don’t know whether the waves exist
  – the particles are temporarily out: difficult to produce and control

• Conclusion:
  – no many options
  – electromagnetism remains the only feasible candidate in the foreseeable future -> you have to learn it!
Electromagnetism = Maxwell

• Maxwell equations:

\[ \nabla \cdot \mathbf{D} = \rho, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]

• Fields in source-free region -> wave equation:

\[ \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \quad \nabla^2 \mathbf{H} - \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \]

• There are 6 field components ("polarization degrees of freedom"). Anyone can be used for communication.
• Only two of them “survive” in free space ("poor" scattering).
Information Theory + Electromagnetism = Spatial Capacity

• Channel model: \( y = Gx + ? \)
• Channel matrix \( G \) is controlled by Maxwell:

\[
G = G(E) \leftarrow \nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0
\]

• Definition of spatial capacity:

\[
S = \max_{p(x), E} \left\{ I(x, \{y, G(E)\}) \right\}
\]

\[
\text{const.: } \langle x^+x \rangle \leq P_T, \quad \nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0, \quad E = E_0 \forall \{r, t\} \in B
\]

• Note 1: max is taken over both \( p(x) \) and \( E \)
• Note 2: conv. power constraint + \( E \)=harmonic function for given boundaries
Another View of Spatial Capacity

- Start with MIMO capacity: \[ C = \log_2 \det \left( I + \frac{\rho}{n} \mathbf{G} \mathbf{G}^+ \right) \]

- Varying the channel \( \mathbf{G} \) varies the capacity.
- Find the maximum!

\[ S = \max_{\mathbf{G}} \left\{ C(\mathbf{G}) \right\}, \text{ const.: } \mathbf{G} \in S(\text{Maxwell}) \]

- Constraint: due to Maxwell, explicit form is unknown
- Additional constraints: limited aperture etc. (practical)

- Does this maximum exists? If so, what is it? What are the main factors that have an impact on it?
Spatial Capacity: Correlation Approach

- How to find the fundamental limit, which is due to the laws of electromagnetism only?
- Get rid of all design-specific details!
- The following assumptions are adopted:
  - limited region of space is considered (similar to limited power)
  - the richest scattering: infinite number of ideal scatterers, uniformly distributed, which do not absorb the EM waves
  - Tx & Rx antenna elements are ideal field sensors, with no size and no mutual coupling
- Capacity is linear in the number of antennas - use as many antennas as possible!
- Is there any limit to this?

$$ C = \log_2 \det \left( I + \frac{\rho}{n} GG^+ \right) \rightarrow C = n \log_2 \left( 1 + \frac{\rho}{n} \right) \rightarrow \frac{\rho}{n} \rightarrow \ln 2 $$
Spatial Capacity: Correlation Approach

- Increasing the number of antennas increases capacity at first.
- Later, one has to reduce antenna spacing to accommodate more antennas within limited space.
- This increases correlation and decreases capacity!
- Some minimum antenna spacing must be respected in order to avoid loss in capacity.
- 2-D analysis shows that this limit is about half a wavelength (Jakes): $d_{\text{min}} \approx \lambda/2$
- 3-D case – roughly the same (sinc)
Equation (1) and (9)

Equation (3) and (9)

Equation (12)

Mean capacity

\[ \Delta = 360^0 \rightarrow d_{\text{min}} = \frac{\lambda}{2} \]
Spatial Capacity: Correlation Approach

- Limited region of space -> limited number of antennas (due to the minimum spacing!)
- Use “sphere packing” argument to estimate it:

\[
\frac{n_{opt}}{V_s} = \frac{6V}{288V} = \frac{6}{288} \frac{V}{\pi \lambda^3} \Rightarrow C_{\text{max}} \approx n_{opt} \log_2 \left(1 + \frac{\rho}{n_{opt}}\right)
\]

- where V is the volume of the space region, \(\rho\) is SNR, and factor 6 is due to 6 “polarizational” degrees of freedom.
- \(C_{\text{max}}\) is the maximum capacity the region of space of volume V is able to provide.
Spatial Capacity: Spatial Sampling Approach

• Antennas just sample the field at various points in space
• Sampling theorem can be used to determine the required number and positions of the antennas
• 3-D Fourier transform in the spatial domain is the key to applying the sampling theorem
• Key difference between temporal and spatial sampling:
  – temporal sampling -> 1-D, $f_{\text{max}}$, sampling interval/rate, no direction
  – spatial sampling -> 3-D, spectrum 2-D boundary (not $f_{\text{max}}$), sampling cell/density, direction is important
• EM field itself posses certain number of degrees of freedom; number of antennas should not be larger
• Information theoretical properties of EM fields
Electromagnetism in Frequency Domain

- Frequency-domain representation:

\[ \phi(r, \omega) = \int \phi(r, t) e^{-j\omega t} dt \quad \Rightarrow \quad \nabla^2 \phi(r, \omega) + \left( \frac{\omega}{c} \right)^2 \phi(r, \omega) = 0 \]

- where \( \phi \) is any of the components of \( E \) or \( H \).

- Plane-wave spectrum expansion:

\[ \phi(k, \omega) = \int \phi(r, \omega) e^{jk \cdot r} dr \quad \Rightarrow \quad \left| k \right|^2 - \left( \frac{\omega}{c} \right)^2 \phi(k, \omega) = 0 \]

\[ \phi(r, t) = \frac{1}{(2\pi)^4} \iiint \phi(k, \omega) e^{j(\omega t - k \cdot r)} dkd\omega \]

- Key observation: the channel matrix entries must satisfy the same wave equation!

\[ g_{ij} = \frac{1}{(2\pi)^3} \int g_j(k, \omega) e^{-j k_i r_i} dk \quad \Rightarrow \quad \left| k \right|^2 - \left( \frac{\omega}{c} \right)^2 g_j(k, \omega) = 0 \]
Plane-Wave Spectrum and Sampling

- Plane-wave spectrum is band-limited, \( k_x, k_y, k_z \leq |k| = \omega / c \)
- This assumes no evanescent waves with imaginary wavenumber
- Apply the sampling theorem. Sampling interval (in each dimension) is
  \[ \Delta x = \Delta y = \Delta z = \frac{\lambda}{2} \]
- The field can be recovered completely from its samples -> no loss of information
- Conclusion: **minimum antenna spacing** is \( \lambda / 2 \) -> this is a fundamental limit!
- For given aperture \( L \) (1-D), \( n_{\text{max}} = \frac{2L}{\lambda} \)
- This limits the capacity according to \( C_{\text{max}} = n \log_2 \left( 1 + \frac{\rho}{n} \right) \)
- The limit is fundamental and is imposed by Maxwell!
- The same limit as for the correlation argument
Correlation and Sampling

- For given angular spread, the correlation and sampling approaches produce roughly the same results!

- Salz-Winters Model: Incoming multipath signals arrive to the linear antenna array within some angle spread ($\pm \Delta$)

\[
\frac{d_{\min}}{\lambda} \approx \max \left\{ \frac{1}{2\Delta}, 0.5 \right\}
\]

\[
d_{\min} = \frac{\lambda}{\Delta}
\]

\[
\Delta = 10^0
\]

\[
\Delta = 1^0
\]

\[
\text{Correlation and Sampling}
\]

\[
\text{Correlation}
\]

\[
\text{Sampling}
\]

\[
\text{Incoming multipath}
\]

\[
\text{Angle spread}
\]

\[
\text{Antenna array}
\]
Some Flaws in The Argument

• Implicit assumption: infinite number of antennas is used (the number of samples must be infinite!)
• May be close to that in temporal domain (i.e., millions of samples) but not in the spatial!
• **Truncation error** must be carefully evaluated
• Some bounds
  - sampling with guard band (over sampling) \( \Delta(t) \leq \frac{4\max\{x(t)\}}{\pi^2 N (1 - r)} \)
  - sampling over finite interval
  - sampling finite energy signal
• Truncation error \( \rightarrow 0 \) as \( N \rightarrow \infty \)
• In terms of capacity? \( \Delta t = T / N \)
• Tx degrees of freedom
Truncation Error and Capacity

- Fix $n_T$ and increase $n_R$ for fixed $L=5$ lambda
- Rich-multipath quasi-static channel

\[ n_{\text{max}} \approx \frac{2L}{\lambda} + 2 \]
2-D and 3-D Sampling

• 1-D antenna -> simple sampling (like temporal)
• 2-D and 3-D cases -> many possibilities, much richer structure
• Minimum spacing is different!
  – 2-D: \( \Delta x = \Delta y = \frac{\lambda}{\sqrt{3}} \)
  – 3-D: \( \Delta x = \Delta y = \Delta z = \frac{\lambda}{\sqrt{2}} \)
• Each additional dimension possesses less degrees of freedom than the previous one
• Rectangular lattice is not optimum!
2-D and 3-D Sampling: Spectral Support

1-D sampling:

$-k_{x,\text{max}}$  $k_{x,\text{max}}$

2-D sampling

Best sampling strategy: depends on the spectral support
Spectral Support of Sampled Signal

Rectangular sampling
\[ \Delta r_{\text{min}} = \lambda / 2 \]

Optimum sampling
\[ \Delta r_{\text{min}} = \lambda / \sqrt{3} \]

\[ \mathbf{v} = \begin{bmatrix} \lambda / 2 & 0 \\ 0 & \lambda / 2 \end{bmatrix} \]

\[ \mathbf{v} = \begin{bmatrix} \lambda / 2 & -\lambda / 2 \\ \lambda / \sqrt{3} & \lambda / \sqrt{3} \end{bmatrix} \]
EM Degrees of Freedom and Quantum Field Theory

- From continuous to discrete variables
- EM field in rectangular volume (a,b,c): \( \mathbf{E}(\mathbf{r}) = \sum_{k} \mathbf{A}_k e^{ikr} \)
- where \( k_x = \frac{2\pi}{a} n_x, \quad k_y = \frac{2\pi}{b} n_y, \quad k_z = \frac{2\pi}{c} n_z \), \( n_x, n_y, n_z \) - integer

- For \( |\mathbf{k}| \leq k_{\text{max}} \), number of degrees of freedom is finite
- Standard approach in quantum electrodynamics: expansion of the field into oscillators (eigenmodes)
- Information capacity of quantum fields?
- Link between information theory and quantum field theory?
Capacities of Waveguide and Cavity Channels: Why?

• Waveguides / cavities can model corridors, tunnels and other confined space channels,
• This is a canonical problem, it allows to develop appropriate techniques, which can be further extended to more complex problems,
• It allows to shed light on the relation between information theory and electromagnetism in most clear form
• the limits imposed by Maxwell on achievable channel capacity follow immediately.
Basic Idea

• Any field inside of a waveguide is a combination of eigenmodes
  \[ E(r) = \sum_k E_k(r) \]

• All eigenmodes are orthogonal (lossless, homogeneous waveguide),
  \[ \int_S E_{\mu} E_{\nu} dS = c \delta_{\mu\nu} \rightarrow G = I \]

• Use the eigenmodes as independent sub-channels!
• MIMO capacity is maximum:
  \[ C = N \log_2 \left( 1 + \frac{\rho}{N} \right) \]
• Need to evaluate N -> electromagnetic analysis
• Lossy/inhomogeneous waveguide -> coupling of eigenmodes. Loss in capacity is low if \( r < 0.5 \)
Waveguide Modes

D. M. Pozar, Microwave Engineering, Wiley
Basic System Architecture

• System architecture is based on the mode orthogonality
• Tx end: all the possible modes are excited (sounds crazy to electromagnetic experts!)
• Rx end: EM field is measured on the cross-sectional area + correlation receiver (with each eigenmode)
• Spatial sampling may be used to reduce the number of field sensors
• Equivalent channel matrix (Tx end-Rx end-correlator output): $G = I$
Rectangular Waveguide

• No evanescent waves:

\[ \gamma_{mn}^2 = \left( \frac{\pi m}{a} \right)^2 + \left( \frac{\pi n}{b} \right)^2 \leq \left( \frac{\omega}{c_0} \right)^2 \]

• This limits the number of modes, i.e. possible \((m,n)\) pairs:

\[ \left( \frac{m \lambda}{a} \right)^2 + \left( \frac{n \lambda}{b} \right)^2 \leq 4 \rightarrow N \approx \frac{2 \pi ab}{\lambda^2} \]

• The number of modes (i.e. channels) is determined by the waveguide cross-section

• This, in turn, limits the capacity -> the limit is fundamental!
Capacity of Rectangular Waveguide

Number of modes in a rectangular waveguide for \( a=b \).

- Exact
- Approximate

\[
\text{Number of modes} \quad \text{vs.} \quad \frac{a}{\lambda}
\]

MIMO capacity of a rectangular waveguide for \( a=b \) and SNR=20 dB.

- Exact
- Approximate
- Limit (12)

\[
\lim_{N \to \infty} C = \frac{\rho}{\ln 2}
\]

one of applications: optics
Capacity of Rectangular Waveguide

- What happens if a linear (1-D) array is used?
- The number of channels decreases!
- OX array: modes are orthogonal if $m_1 \neq m_2$
- OY array: modes are orthogonal if $n_1 \neq n_2$
- Number of modes:
  \[ N_{xy} \approx \frac{2\pi ab}{\lambda^2} \rightarrow N_x \approx \frac{4a}{\lambda} \text{ or } N_y \approx \frac{4b}{\lambda} \]
- Maximum “reasonable” number of antennas:
  \[ C = \frac{\rho}{\ln 2} \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} \left( \frac{\rho}{N} \right)^i \approx \frac{\rho}{\ln 2} \left( 1 - \frac{\rho}{2N} \right) \rightarrow N_{\text{max}} \approx \rho \]
- Max. “reasonable” size:
  \[ \frac{a_{\text{max}}}{\lambda} \approx \sqrt{\frac{\rho}{2\pi}} \text{ (2-D array), } \frac{a_{\text{max}}}{\lambda} \approx \frac{\rho}{4} \text{ (1-D OX array)} \]
Circular Waveguide

- Similar approach can be used

\[ \gamma_{mn} = \frac{p_{mn}}{a} \text{ (E mode), } \gamma_{mn} = \frac{p'_{mn}}{a} \text{ (H mode)} \leq \frac{\omega}{c_0} \]

- where \( p \) and \( p' \) are the roots of Bessel functions and their derivatives, \( a \) is the radius

- The number of modes is limited by

\[ p_{mn} \leq 2\pi a / \lambda \text{ (E modes), } p'_{mn} \leq 2\pi a / \lambda \text{ (H modes)} \]

- and, using wavenumber space filling, approximately

\[ N \approx \frac{10a^2}{\lambda^2} \]
Some Remarks

• Compare rectangular and circular waveguides:

\[ N_{xy} \approx \frac{2\pi ab}{\lambda^2} \rightarrow N_x \approx \frac{4a}{\lambda} \text{ or } N_y \approx \frac{4b}{\lambda} \]

\[ N_{cir} \approx \frac{10a^2}{\lambda^2} \]

• The number of modes is determined by the cross-section (in terms of wavelength)

• **Conjecture**: this is true for arbitrary cross-section,

\[ N_{arb} \sim S/\lambda^2 \]

• In all cases, this corresponds to sampling at \( \Delta r \sim \lambda/2 \)
• Structure of EM field has a profound impact on capacity!
Rectangular Cavity

- Eigenmodes exist at certain frequencies only:
  \[ k^2 = \left( \frac{\pi m}{a} \right)^2 + \left( \frac{\pi n}{b} \right)^2 + \left( \frac{\pi p}{c} \right)^2 = \left( \frac{\omega}{c_0} \right)^2 \]

- Consider a narrow band of frequencies and find the number of eigenmodes for \( k \in [k_0, k_0 + \Delta k] \):
  \[ N_c \approx \frac{8 \pi V_c}{\lambda^3} \frac{\Delta f}{f_0} \]

- Orthogonality: \( \int_0^V \int E_\mu E_\nu dV = c \delta_{\mu \nu} \)

- Reduced 2-D version: modes with different (m,n) are orthogonal

- Critical length: \( c > c_t = \frac{f_0 \lambda}{4 \Delta f} \rightarrow N_c \approx N_w \)

- Long cavity is the same as waveguide
Capacity of Rectangular Cavity

Number of orthogonal modes

Capacity in a rectangular cavity

\[ a = 4\lambda, \quad b = 2\lambda \quad \Delta f / f_0 = 0.01 \]
Conclusions

• Information theory and electromagnetism: a fundamental link exists
• Future unification is inevitable
• Maxwell limits MIMO capacity
• Half a wavelength is a fundamental limit
• EM field has a finite number of degrees of freedom
• MIMO capacity in confined spaces – eigenmode analysis
• MIMO capacity in open spaces – spatial sampling
• New insights into waveguide performance
• Waveguide has a limited capacity! (by both IT and EM)